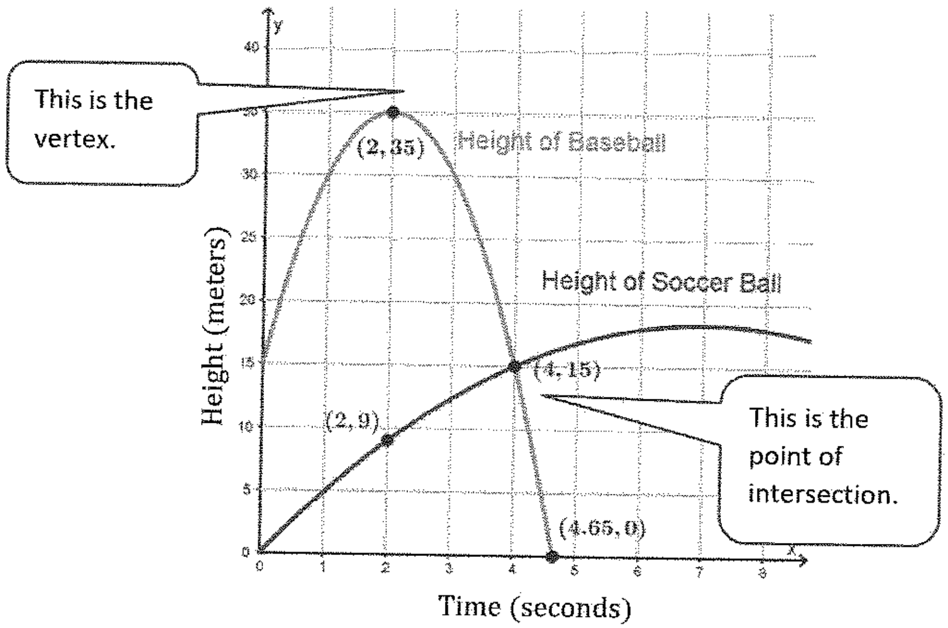
Lesson 1: Analyzing a Graph

Jane throws a baseball from a second floor window. At the same time, Kevin kicks a soccer ball from the ground below the window. The graphs of the height of each ball in terms of time t seconds since Jane threw the baseball are shown below.



a. According to the graphs, what type of function is being modeled?

The height of the baseball appears to be modeled by a quadratic function. The height of the soccer ball appears to be modeled by a quadratic function as well.

b. What are some key features of the graphs of these functions?

The vertex of the baseball graph is $(2, 35)$, and the x-intercept point is $(4.65, 0)$. The y-intercept point of the baseball graph is $(0, 15)$. Both graphs open downward, which means the vertices are maximum points. The soccer ball graph includes the origin. The vertex of the soccer ball graph appears to be near the point $(7, 18)$.

c. Will the two balls ever be at the same height at the same time? If so, when and at what height?

The intersection point has coordinates $(4, 15)$. The two balls will be at the same height, 15 meters, at 4 seconds.

d. Create a function to model the height of the baseball. Use the data points labeled on the graph to create a precise model for the ball's height.

Use the vertex form of a quadratic function to write the equation.

$f(x) = a(x - h)^2 + k$

From the graph, the vertex is $(2, 35)$*, so* $h = 2$ *and* $k = 35$*. Use the point* $(4, 15)$ *to substitute into the function to find* a*.*

$$15 = a(4 - 2)^2 + 35$$
$$15 = 4a + 35$$
$$-20 = 4a$$
$$\frac{-20}{4} = \frac{4a}{4}$$
$$-5 = a$$

The value of a is negative, which means the graph of the function will open downward.

$$f(x) = -5(x - 2)^2 + 35$$

e. Which ball is traveling faster on the interval $[2, 4]$? Show work to support your answer.

Baseball:

$$\frac{f(4) - f(2)}{4 - 2} = \frac{15 - 35}{4 - 2} = \frac{-20}{2} = -10$$

Soccer ball:

$$\frac{g(4) - g(2)}{4 - 2} = \frac{15 - 9}{4 - 2} = \frac{6}{2} = 3$$

I find the average rate of change for both functions to estimate which ball is traveling faster from 2 to 4 seconds. The negative sign tells me whether the ball is moving up or down, which does not affect the speed.

On the time interval $[2, 4]$*, the average rate of change for the baseball is* -10 *meters per second, and the average rate of change for the soccer ball is* 3 *meters per second. Therefore, the baseball is traveling faster than the soccer ball during this time interval.*

Lesson 2: Analyzing a Data Set

A Quadratic Data Set

1. Use the table shown to the right to answer the following questions.

x	y
-3	21
-2	12
-1	5
0	0
1	-3
2	-4
3	-3

a. Determine the function type that could be used to model the data set, and explain why.

The first differences are not constant which means the function is not linear. The second differences are constant which means this data set could be modeled by a quadratic function.

First Differences:

$12 - 21 = -9$

$5 - 12 = -7$

$0 - 5 = -5$

Second Differences:

$-7 - (-9) = 2$

$-5 - (-7) = 2$

The first differences are increasing by 2 each time x increases by 1.

This is the minimum value for the function because of the symmetry of the graph of a quadratic function. I can see that $x = 2$ is the axis of symmetry.

b. Complete the data set using the patterns you described in part (a).

If the first differences are increasing by 2 and this pattern continues, then the differences in order would be $-9, -7, -5, -3, -1, 1, 3$, etc. Thus to get the y-value when $x = 3$, add 1 to -4. The missing value is -3.

c. If it exists, find the minimum or maximum value for the function model. If there is no minimum or maximum, explain why.

Minimum point is $(2, -4)$. The minimum value is -4.

I recall that the minimum or maximum value is just the output value.

2. Use the table shown to the right to answer the following questions.

a. Determine the function type that could be used to model the data set, and explain why.

The first differences show that the function is linear with a constant difference of -2.

First Differences:

$9 - 11 = -2$

$7 - 9 = -2$

x	y
-3	11
-2	9
-1	7
0	5
1	3
2	1
3	-1

The y-values are decreasing by 2 units for every 1 unit increase in the x-values.

I can fill in the missing y-values by subtracting 2 from the previous y-value.

b. Complete the data set using the special pattern of the function you described in part (a).

If the first difference is -2, then the missing values are 5, 3, and -1.

c. If it exists, find the minimum or maximum value for the function model. If there is no minimum or maximum, explain why.

There is no minimum or maximum value for a linear function with a nonzero slope whose domain is all real numbers. For this function, the y-values decrease as the x-values increase for all x-values in the domain of the function.

Lesson 3: Analyzing a Verbal Description

This a uniform rate. I know these situations are modeled by linear functions. The rate is the slope.

A Verbal Description for a Linear Model

1. Dan started running 2 miles from his house at a constant speed of 6 mph on a straight road.
   a. What type of function would best model Dan's distance from his house while he is running?

      A linear function would be the best model because his distance is changing at a constant rate.

   b. What function represents Dan's distance from his house in miles as a function of the number of hours he has been running?

      $f(x) = 2 + 6x$ where $f(x)$ is the distance in miles from Dan's house after x hours of running since he was 2 miles from his house.

      I need to explain the meaning of my variables.

   c. If Dan wants to run to his friend's house, which is 10 miles down the road from his house, how long will this take if he maintains his constant speed?

      Solve the equation $f(x) = 10$.

      $f(x)$ represents the distance from Dan's house in miles after running for x hours since he was 2 miles from his house, so I need to make it equal to 10.

      The solution to $2 + 6x = 10$ is $x = \frac{4}{3}$.

      It will take Dan $1\frac{1}{3}$ hours, or 1 hour and 20 minutes, to get to his friend's house.

A Verbal Description for an Exponential Model

2. Eduardo works for a technology start-up. During the first 24 months, the revenue increases by 5% per month. At the end of the first month, the technology start-up has $2,500 in revenue.
   a. What type of function would best model the revenue at Eduardo's company?

      An exponential function would be the best model because each month the revenue increases by a percentage of the previous month's revenue.

      A percent increase or decrease indicates an exponential function.

b. What function represents the revenue at Eduardo's company as a function of the number of months since the business started?

$f(n) = 2500(1.05)^{n-1}$

where $f(n)$ is the revenue at the end of the n^{th} month.

I use $n - 1$ because the first month must equal 2,500 when I substitute 1 for n.

c. How much revenue was earned by the end of the 24th month?

Evaluate $f(24)$.

$f(24) = 2500(1.05)^{24-1} \approx 7678.81$

The company earned approximately $\$7,678.81$ in revenue by the end of the 24^{th} month.

When an object hits the ground, the height is 0 ft.

A Verbal Description that Leads to a Quadratic Model

3. A model rocket is launched from a 20-foot platform upward into the air. It hits the ground 8 seconds later.

a. What type of function represents this situation?

A quadratic function represents the situation because it is describing projectile motion.

b. What was the initial velocity of the rocket?

The function, $f(t) = -16t^2 + v_0t + s_0$ where v_0 is the initial velocity in feet per second and s_0 is the initial height in feet, models the height of a projectile in feet after t seconds.

I learned about this function in Module 4. Use -16 when measuring the height in feet.

$$0 = -16(8)^2 + v_0(8) + 20$$
$$0 = -16(64) + 8v_0 + 20$$
$$1024 - 20 = 8v_0$$
$$\frac{1004}{8} = v_0$$

I substitute the values I know into the equation and solve for v_0.

The initial velocity v_0 is 125.5 ft/sec.

c. What function models this situation?

$f(t) = -16t^2 + 125.5t + 20$ *where $f(t)$ is the height in feet above ground after t seconds.*

d. What is the maximum height attained by the rocket?

The x-coordinate of the vertex of a quadratic function of the form $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

$a = -16$ and $b = 125.5$, so $x = -\frac{125.5}{2(-16)} = 3.921875$

The maximum height is $f(3.921875)$.

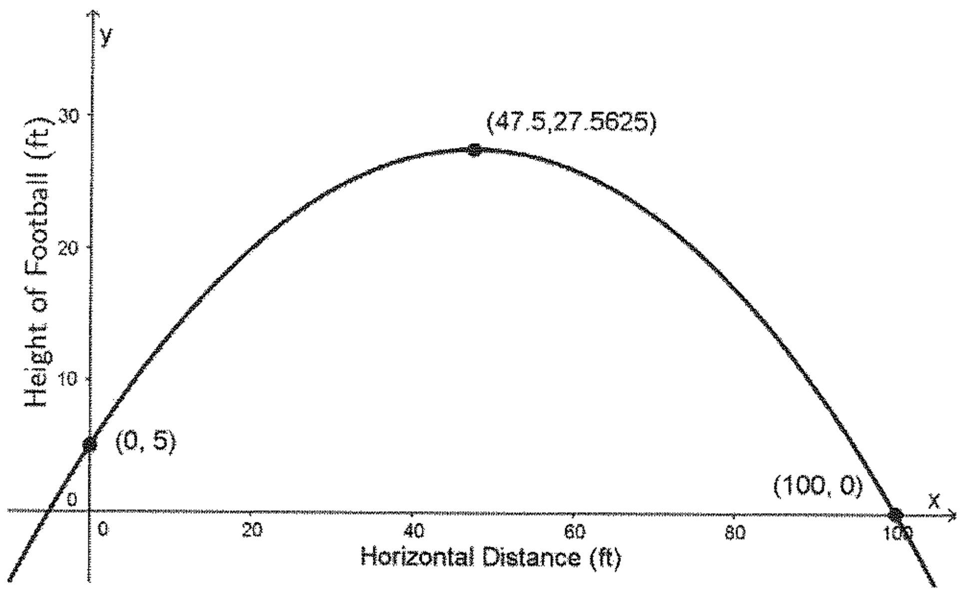
$f(3.921875) \approx 266.1$

The maximum height is about 266.1 feet.

Lesson 4: Modeling a Context from a Graph

Writing an Algebraic Function from a Graph

The graph shown below models the height and the horizontal distance a football has traveled since it was released by the quarterback.



1. Write a quadratic function to model the path of the football.

The vertex is $(47.5, 27.5625)$. *Use the vertex form of a quadratic function to write the equation.*

$$f(x) = a(x - h)^2 + k$$
$$f(x) = a(x - 47.5)^2 + 27.5625$$

Use the coordinate point $(0, 5)$ *to determine the value of* a.

Substitute 0 *for* x *and* 5 *for* $f(x)$, *and solve for* a.

$$5 = a(0 - 47.5)^2 + 27.5625$$
$$5 = a(-47.5)^2 + 27.5625$$
$$-22.5625 = 2256.25a$$
$$-0.01 = a$$

When I know the vertex and at least one other point on the graph of a quadratic function, I can write the function in vertex form.

The quadratic function is $f(x) = -0.01(x - 47.5)^2 + 27.5625$.

2. How high was the football when it had traveled a horizontal distance of 20 feet?

Evaluate $f(20)$.

$$f(20) = -0.01(20 - 47.5)^2 + 27.5625 = 20$$

When the football had traveled 20 feet horizontally, it was 20 feet in the air.

3. How far had the football traveled horizontally when its height was 25 feet?

Solve the equation $f(x) = 25$.

$$25 = -0.01(x - 47.5)^2 + 27.5625$$
$$-2.5625 = -0.01(x - 47.5)^2$$
$$256.25 = (x - 47.5)^2$$
$$\sqrt{256.25} = x - 47.5 \text{ or } -\sqrt{256.25} = x - 47.5$$
$$x = 47.5 + \sqrt{256.25} \text{ or } x = 47.5 - \sqrt{256.25}$$

I need to subtract 27.5625 from both sides of the equation and divide both sides by -0.01.

When I take the square root, there will be 2 solutions.

The approximate solutions are 63.5 and 31.5.

When the football is 25 feet in the air, it has traveled approximately 31.5 feet and 63.5 feet horizontally from where it was thrown.

I can see from the graph that there will be two horizontal distances that correspond to a height of 25 feet.

4. Suppose the football on average traveled 22 feet horizontally for every second it was in the air. How long was the football in air?

This is a proportional relationship.

$$22t = 100$$
$$t = \frac{100}{22}$$

I need to divide 100 feet, the total horizontal distance, by 22 feet/second.

The solution to this equation is $4.\overline{54}$. That means it takes about 4.5 seconds for the football to travel horizontally 100 feet.

Lesson 5: Modeling from a Sequence

Find a function/formula that represent the n^{th} term of the sequence described in each situation.

Modeling a Situation with an Arithmetic Sequence

Maria is starting a new running program. She decides to increase her weekly mileage by 0.5 miles per week.

1. Maria starts by running 6 miles during the first week. Complete the table below to show her total mileage each week.

Week	Total Miles for the Week
1	6
2	6.5
3	7
4	7.5
5	8
6	8.5

2. Write a function that models Maria's total miles for each week as a function of the number of weeks she has been running.

   $f(n) = \textbf{initial distance} + (n-1)(\textbf{common difference})$ *where $f(n)$ is the total number of miles for the week after n weeks.*

   The initial distance is 6 miles. The common difference is 0.5 miles. The function is

   $$f(n) = 6 + (n-1) \cdot 0.5.$$

3. Suppose Maria usually runs 4 times per week. What is her average distance for a single run during the tenth week?

   During the tenth week, she will run 10.5 miles.

   $$f(10) = 6 + (10-1) \cdot 0.5$$
   $$f(10) = 10.5$$

   Dividing 10.5 by 4 gives an average distance for a single run of 2.625 miles.

EUREKA MATH

Modeling a Situation with a Geometric Sequence

Molly is also starting a running program. She decides to increase her weekly mileage by 25% each week.

4. Molly starts by running 5 miles during the first week. Complete the table below to show her mileage each week. Round the number of miles to the nearest hundredth.

Week	Total Miles for the Week
1	5
2	6.25
3	7.81
4	9.77
5	12.21
6	15.26

Total miles is increasing by a factor of 1.25 per week. I can use a calculator to get more precise answers, and then round them to the nearest hundredth.

5. Write a function that models Molly's total miles for each week as a function of the number of weeks she has been running.

$f(n) = \text{initial distance}(\text{common ratio})^{n-1}$ *where* $f(n)$ *is the total number of miles for the week after* n *weeks.*

The initial distance is 5 *miles. The common ratio is* 1.25*. The function is*

$$f(n) = 5(1.25)^{n-1}.$$

I need to subtract 1 from n to make the function match the table values.

6. How far will Molly run during the tenth week if she maintains this schedule?

During the tenth week, she will run approximately 37.25 *miles.*

$$f(10) = 5(1.25)^{10-1}$$
$$f(10) \approx 37.25$$

7. Does this mileage increase seem like a realistic long-term plan? Explain your reasoning.

Student answers will vary but should state that this is not a realistic schedule for the long term. If she runs 4 *times per week, she will be running approximately* 9 *miles on each run.*

Modeling a Situation That is Not Arithmetic or Geometric

Mikey also started a running program. His total mileage each week is shown in the table below.

Week	Total Miles for the Week
1	2
2	5
3	10
4	17
5	26
6	37

There is a linear pattern in the common differences.

$5 - 2 = 3$

$10 - 5 = 5$

$17 - 10 = 7$

$26 - 17 = 9$

8. Write a function that models Mikey's total miles for the week as a function of the number of weeks he has been running.

When the common differences have a linear pattern, the sequence can be modeled by a quadratic function.

The function $f(n) = n^2$ does not work because $f(1) = 1$ and $f(2) = 4$ for this function, and the table requires the function outputs to be 2, 5, 10, etc. Try the function $f(n) = n^2 + 1$.

The total miles are one greater than the square numbers, 1, 4, 9, 16, and so on.

$f(1) = 1^2 + 1 = 2$
$f(2) = 2^2 + 1 = 5$
$f(3) = 3^2 + 1 = 10$

I need to check this function to see if it gives the correct values.

The function $f(n) = n^2 + 1$ gives the correct values for the rest of the table as well.

The function that models Mikey's total miles for each week is $f(n) = n^2 + 1$ where $f(n)$ is the total number of miles for the week after n weeks.

EUREKA MATH

Lesson 6: Modeling a Context from Data

Lesson Notes

Students need access to the Internet to complete the problem set for this lesson. They should research and find three situations that resemble the functions, tables, and graphs shown below.

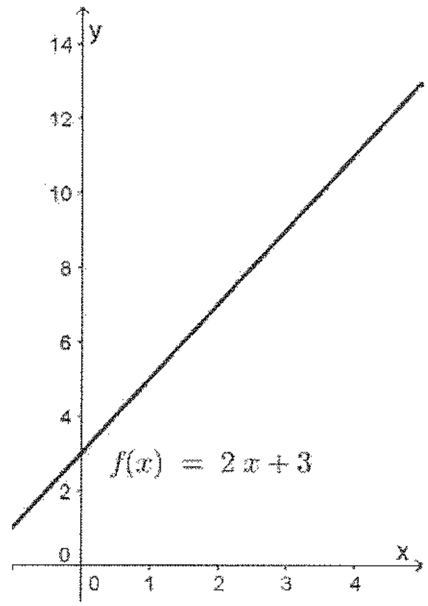
A Linear Function

This situation is modeled by a graph that is a line. The function is of the form $f(x) = ax + b$ where a is the slope of the line, and b is the y-intercept of the line.

FUNCTION: $f(x) = 2x + 3$

TABLE:

x	$f(x)$
0	3
1	5
2	7
3	9



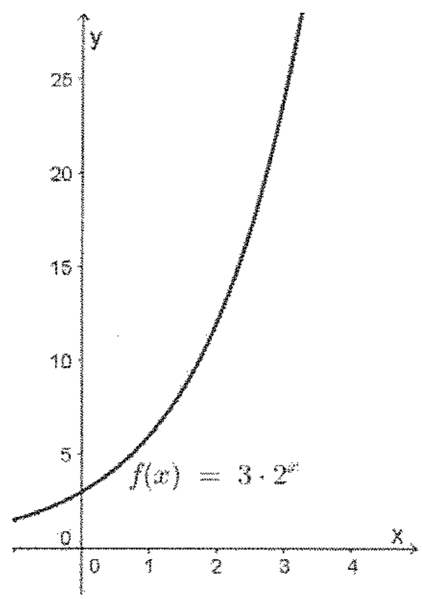
An Exponential Function

This situation is modeled by a curved shaped graph. The function is of the form $f(x) = a \cdot b^x$ where a is the y-intercept of the graph and b is the constant ratio of increase or decrease.

FUNCTION: $f(x) = 3 \cdot 2^x$

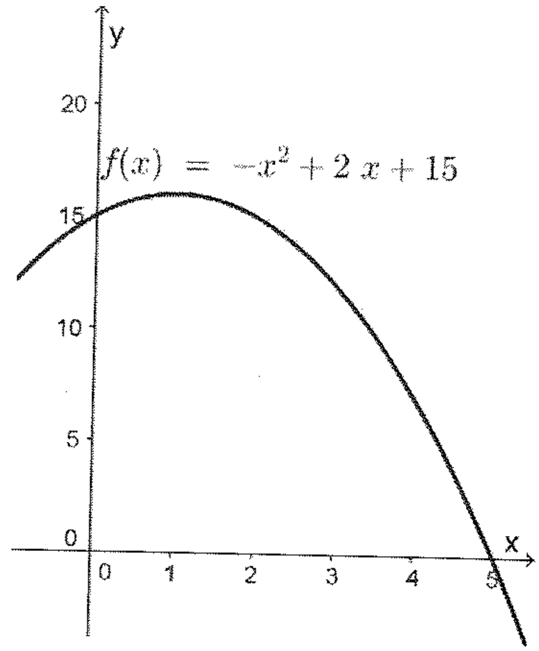
TABLE:

x	$f(x)$
0	3
1	6
2	12
3	24



A Quadratic Function

This situation is modeled by a u-shaped curve. The function is of the form $f(x) = ax^2 + bx + c$. When the inputs (x-values) increase by equal amounts, the first differences form a linear pattern. The second differences are constant.



FUNCTION:

$$f(x) = -x^2 + 2x + 15$$

TABLE:

x	$f(x)$
0	15
1	16
2	15
3	12

EUREKA MATH

Lesson 7: Modeling a Context from Data

Lesson Notes

Students need a graphing calculator or access to web-based applications such as desmos.com to create scatter plots and regression equations. Homework Helpers for Algebra I Module 2 Lessons 12–19 include detailed directions for using a TI-84 graphing calculator to find regression equations. On the TI-84, the quadratic regression equation is found by pressing STAT, selecting CALC, and then 5:QuadReg.

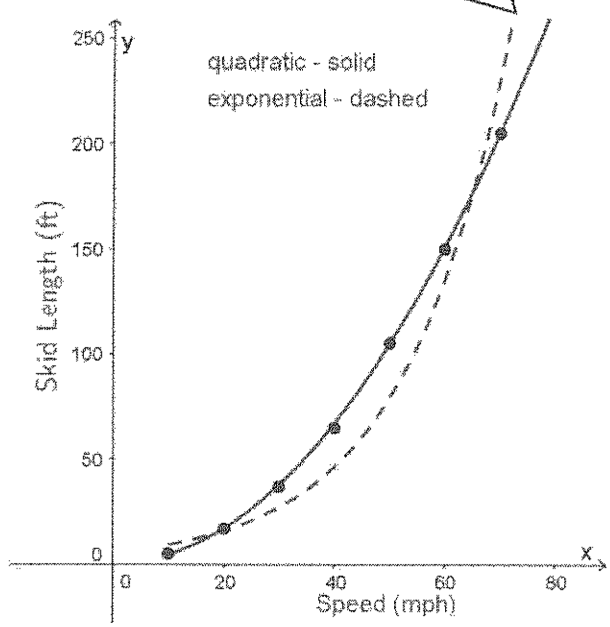
Students can analyze the rough patterns in the differences or ratios of consecutive terms to decide whether a linear, exponential, or quadratic equation would be most appropriate.

Data Modeled by a Quadratic Equation

When police investigate the scene of an automobile accident, they can use the length of the skid mark to estimate the speed of the car. The results of experiments with a test car are shown in the table below.

Speed (mph)	Skid Length (ft)
10	5
20	17
30	37
40	65
50	105
60	150
70	205
80	265

I can make a scatter plot. Based on the scatter plot, I need to use a nonlinear equation to model the situation. The quadratic regression equation fits the data better than an exponential regression equation.



1. What is the best model for this data? Explain your reasoning.

   A quadratic model makes the most sense. The scatter plot is clearly not linear. An exponential model does not make sense because there is not a constant quotient for successive skid lengths.

   $\frac{17}{5} = 3.4$

   $\frac{37}{17} \approx 2.18$

   $\frac{65}{37} \approx 1.76$

The quadratic regression equation, with the values of a, b, and c rounded to the nearest thousandth, is $f(x) = 0.042x^2 - 0.018x + 0.554$, where $f(x)$ models the skid length in feet of a car traveling x miles per hour when it hits the brakes. From the graphing calculator, $r^2 = 0.99988$, which indicates that this is a good model for the given data.

2. Based on your model, what is the skid length if the car is traveling 55 mph? 100 mph?

Evaluate $f(55)$ and $f(100)$.

$f(55) \approx 125.8$

$f(100) \approx 416.0$

I can use the regression equation stored on my calculator to quickly evaluate the function at $x = 55$ and $x = 100$. Using the stored regression equation gives more accurate results. If I evaluate $f(55)$ and $f(100)$ using rounded values for the coefficients a, b, and c, then I will get a slightly different answer.

At 55 mph, the function predicts a skid length of approximately 125.8 ft. At 100 mph, the function predicts a skid length of approximately 416.0 ft.

3. For this model, would it make sense to have an input value (speed) greater than 150 mph? A negative speed?

Most cars that people drive for everyday use have a maximum speed of 150 mph or less, so higher speeds would not make sense. Speed is a positive quantity, so negative speeds also do not make sense in this situation.

In this situation, only positive domain values up to approximately 150 make sense.

EUREKA MATH

Lesson 8: Modeling a Context from a Verbal Description

Modeling with an Exponential Function from a Verbal Description

1. In humans, the half-life of caffeine, the time required by the body to metabolize half of the substance, ranges from 4 to 6 hours. Suppose Daphne drinks a cup of coffee that contains 150 mg of caffeine. Assume that for Daphne, the half-life of caffeine is 5 hours.

   a. How much caffeine remains in her bloodstream 10 hours after she drinks the coffee?

Time (hrs)	*Caffeine in Bloodstream* (mg)
0	150
5	75
10	37.5
15	18.75

I am going to create a table of values to find the answer.

I see that an exponential function should be used to model this situation because there is a constant quotient for any two successive output values.

10 hours after drinking the coffee, 37.5 mg of caffeine remains in her bloodstream.

   b. Create a formula that models the amount of caffeine, in mg, in Daphne's bloodstream t hours after she drinks the coffee.

Let $A(t)$ represent the amount of caffeine in Daphne's bloodstream t hours after she drinks the coffee.

$$A(t) = 150\left(\frac{1}{2}\right)^{\frac{t}{5}}$$

I know that every 5 hours the amount of caffeine is multiplied by $\frac{1}{2}$. This means I need to divide t by 5.

I can check my formula using the values from my table.

$$A(5) = 150\left(\frac{1}{2}\right)^{\frac{5}{5}} = 150\left(\frac{1}{2}\right) = 75$$

$$A(10) = 150\left(\frac{1}{2}\right)^{\frac{10}{5}} = 150\left(\frac{1}{2}\right)^{2} = 37.5$$

Modeling with a Quadratic Function from a Verbal Description

2. The math club decides to make and sell t-shirts as a fund-raiser. The members of the math club estimate that the revenue can be represented by the function $R(x) = 100x - 0.25x^2$ and that the cost can be represented by the function $C(x) = 5000 + 20x$ where x is the number of t-shirts. How many t-shirts should they make and then sell in order to yield the maximum profit? What is the maximum profit the club can earn?

Let $P(x)$ represent the profit if the club makes and then sells x t-shirts.

$P(x) = R(x) - C(x)$

$P(x) = 100x - 0.25x^2 - (5000 + 20x)$

$P(x) = -0.25x^2 + 80x - 5000$

> I know that profit = revenue − cost. I can use this relationship to write the profit function.

> In order to find the maximum profit, I need to find the vertex of the graph of $y = P(x)$. I can complete the square to rewrite this function in vertex form.

$P(x) = -0.25(x^2 - 320x \quad) - 5000$

$P(x) = -0.25(x^2 - 320x + 25600 - 25600) - 5000$

$P(x) = -0.25(x^2 - 320x + 25600) + 6400 - 5000$

$P(x) = -0.25(x - 160)^2 + 1400$

When $x = 160$, the profit function is at its maximum, so the club should make and sell 160 t-shirts in order to maximize profit. The maximum profit is $\$1,400$.

> I can use the vertex, $(160, 1400)$, to answer the question.

Lesson 9: Modeling a Context from a Verbal Description

An Exponential Growth Model

In 1987, a popular coffee shop began expanding to 17 locations worldwide. From 1987 to 2000, the number of locations for this coffee shop increased by approximately 52.2% per year.

1. Create a formula that models the number of coffee shop locations t years since 1987.

   Let S be the function that models the number of coffee shops t years since 1987.

   $S(t) = 17(1 + 0.522)^t = 17(1.522)^t$

   Since this is an example of exponential growth, I know that I need to write a formula in the form $S(t) = a \cdot b^t$ where a is the initial value and b is the growth factor.

2. Using your model, complete the table below by finding the predicted number of coffee shop locations for each year in the table.

Year	Predicted Number of Locations	Actual Number of Locations
1995	490	677
2000	$3,998$	3,501
2002	$9,261$	5,886
2006	$49,697$	12,440
2013	$940,221$	19,767

I recall that t represents years since 1987. So 1987 corresponds to $t = 0$.

$S(8) = 17(1.522)^8 \approx 490$

$S(13) = 17(1.522)^{13} \approx 3998$

$S(15) = 17(1.522)^{15} \approx 9261$

$S(19) = 17(1.522)^{19} \approx 49697$

$S(26) = 17(1.522)^{26} \approx 940\,221$

To find the predicted number of locations in 1995, I need to evaluate $S(8)$.

3. Based on these values in the table, what would you say about the validity of the formula created in Problem 1?

Examining the table, I see that after the year 2000 the difference between the projected and actual number of locations gets larger.

The formula appears to be fairly accurate for the years 1987 to 2000. After that, the formula predicts much larger numbers than the actual values provided. It appears that after 2000 the number of locations continued to increase but at a slower rate. Therefore, the formula is only valid for a domain of $0 \leq t \leq 13$.