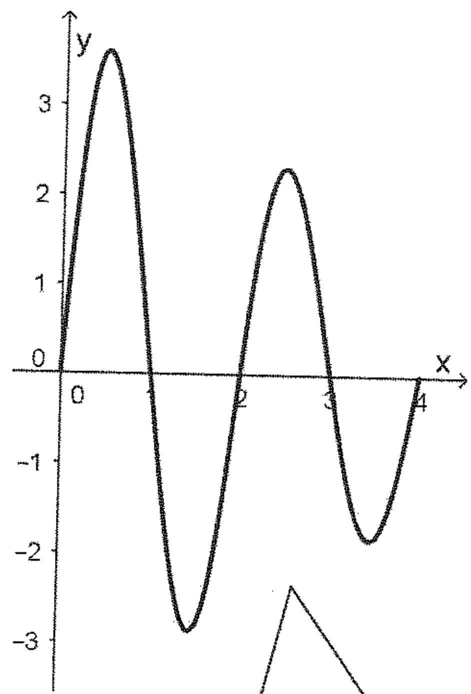
Lesson 13: Interpreting the Graph of a Function

Lesson Notes

Students model the trajectory of the Mars Curiosity Rover. The first few questions in the Problem Set complete the modeling cycle, and students' solutions will vary depending on what happened in class. The rest of the exercises help students to practice interpreting the features of the graph of a function.

The graph of the function f is shown to the right.



I know that the intercepts are where the graph of the function intersects the axes.

Intercepts

1. Identify the x-intercepts for the graph of the function.

   The x-intercepts are 0, 1, 2, 3, and 4.

2. Identify the y-intercept for the graph of the function.

   The y-intercept is 0.

I need to remember that parentheses can mean an interval or an ordered pair. Here it means an interval of x-values.

Positive intervals are where the graph is above the horizontal axis. Negative intervals are where the graph is below the horizontal axis.

Positive and Negative Intervals

3. Identify the positive intervals for the graph of the function.

   $(0, 1)$ *and* $(2, 3)$

4. Identify the negative intervals for the graph of the function.

   $(1, 2)$ *and* $(3, 4)$

Increasing Decreasing Intervals

5. Identify the decreasing intervals for the graph of the function.

   $\left[\frac{1}{2}, \frac{3}{2}\right]$ *and* $\left[\frac{5}{2}, \frac{7}{2}\right]$

6. Identify the increasing intervals for the graph of the function.

   $\left[0, \frac{1}{2}\right]$, $\left[\frac{3}{2}, \frac{5}{2}\right]$, *and* $\left[\frac{7}{2}, 4\right]$

   I will have to estimate the x-values that define each interval. I learned about intervals and interval notation in Module 1.

Lesson 14: Linear and Exponential Models—Comparing Growth Rates

Lesson Notes

Students learned to create linear models given two points in Grade 8 and Module 1 and to use technology in Module 2. Finding a formula for an exponential function given only two points can be tricky unless the growth rate is given or easily obtained. When finding the growth rate is difficult, technology can help. The steps to find an exponential regression equation on a Texas Instruments graphing calculator can be found at the end of this lesson's guide. Free, web-based graphing applications such as www.desmos.com can also be used.

Exponential Growth Rates

1. The function $f(x) = 100(1.4)^x$ models the number of subscribers to a music listening service in x years since the service was launched.

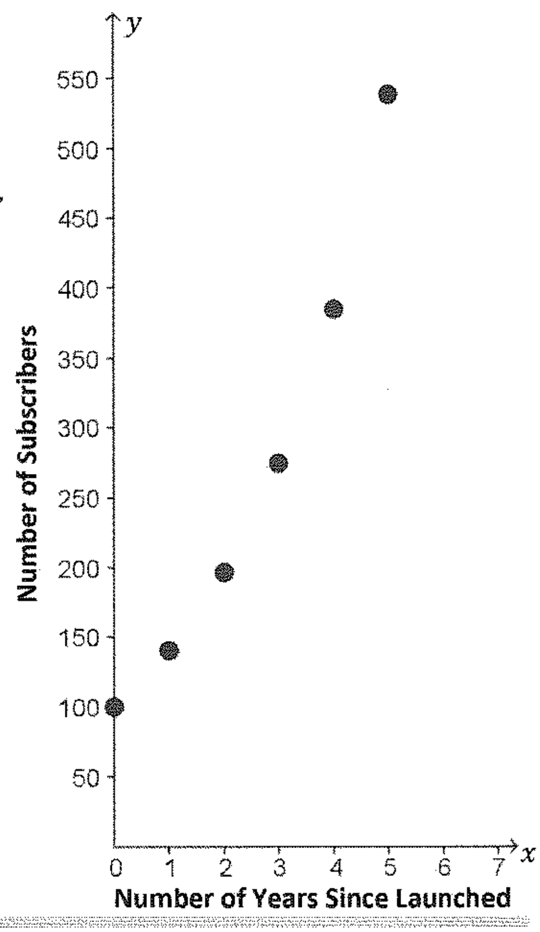
   a. How many times greater are the subscribers at the end of the 10^{th} year compared to at the end of the 5^{th} year?

   To find out how many times greater $f(10)$ is than $f(5)$, calculate $\frac{f(10)}{f(5)}$.

   $$\frac{f(10)}{f(5)} = \frac{100(1.4)^{10}}{100(1.4)^5} = 1.4^{10-5} = 1.4^5 \approx 5.38$$

   b. Graph the points $(x, f(x))$ for x-values 0, 1, 2, 3, 4, and 5.

   I need to evaluate the function at each x-value to find the y-coordinates of each point. Then, I plot the set of ordered pairs. I know $f(0) = 100(1.4)^0 = 100$, so I plot the point $(0,100)$.



c. What is the meaning of the point $(0, f(0))$?

$x = 0$ corresponds to the time the service was launched. The service started with 100 subscribers.

I need to compare consecutive differences and consecutive ratios to help me choose between a linear and an exponential function.

2. The following table shows a population of foxes over a period of 4 years.

Years Since 2010	0	1	2	3	4
Number of Foxes	520	416	333	266	213

a. Write a function to model this population.

The consecutive differences are not constant because $416 - 520 \neq 333 - 416$. The consecutive factors are all very close to 0.8 because $\frac{416}{520} = 0.8$ and $\frac{333}{416} \approx 0.8$.

Use an exponential model, $f(x) = 520(0.8)^x$, where $f(x)$ is the number of foxes, and x is the number of years since 2010.

I can use the formula $f(x) = a \cdot b^x$ to model exponential decay.

b. Estimate when the population will fall below 100 foxes.

Rounded to the nearest whole number,

$$f(6) = 136$$
$$f(7) = 109$$
$$f(8) = 87$$

I need to find when $f(x)$ goes below 100.

Between 7 and 8 years after 2010, the population will fall below 100.

I can represent each plan using a sequence, which I studied in Topic A.

Compare Linear and Exponential Growth Rates

3. Two companies offer the following stock options for employees.
   - Company A: 500 shares to start and an additional 10 shares per month.
   - Company B: 50 shares to start, with the number of shares granted increasing by 10% per month.

   a. Which company will have granted the most shares at the end of 2 years?

   After 2 years, Company A will have awarded 10 additional shares a total of 24 times. The shares at the end of each month can be modeled by the arithmetic sequence

   $$\{500, 500 + 10, 500 + 10(2), 500 + 10(3), \ldots, 500 + 10(24)\}.$$

   $500 + 10(24) = 740$

   There will be 740 shares.

   After 2 years, Company B will have awarded a different number of shares each month. The shares at the end of each month can be modeled by the geometric sequence

   $\{50, 50(1.1), 50(1.1)^2, 50(1.1)^3, \ldots, 50(1.1)^{24}\}$.

   There will be 492 shares.

   I calculated the value of $50(1.1)^{24}$ and rounded to nearest whole number.

   b. Which company will have granted the most shares at the end of 5 years?

   Five years is 60 months. Compare the values of $500 + 10(60)$ and $50(1.1)^{60}$.

   $$500 + 10(60) = 1100$$

   $$50(1.1)^{60} \approx 15224$$

   Company B will have granted the most shares.

   c. In your opinion, which company offers the better stock option for its employees?

   Student answers will vary but should mention that initially, the linear growth option offered by Company A will yield more shares, but eventually, the exponential growth option offered by Company B will yield more. If you plan to stay fewer than 2 years, definitely stick with Company A to earn the most shares; but if you plan to stay for 5 or more, then Company B is the better option.

The points will be of the form (years since 1920, population in millions).

Compare Linear and Exponential Models Given Two Points

4. According to the U.S. Census (source: www.census.gov), the United States population in 1920 was approximately 106 million. In 1990, the population was 249 million.

If I let x represent the years since 1920, then 106 will be the y-intercept in both models.

I can use $f(x) = mx + b$, where m is the slope, and b is the y-intercept.

a. Write a linear function to model the U.S. population in millions of people.

$m = \frac{249-160}{70} \approx 1.27$. *The linear function* $f(x) = 1.27x + 106$ *will give the population in millions, where* x *is the number of years since 1920.*

b. Write an exponential function to model the U.S. population in millions of people.

Using an exponential regression on the graphing calculator, $g(x) = 106(1.0123)^x$ *will give the population in millions, where* x *is the number of years since 1920.*

I can use a graphing calculator to get a function.

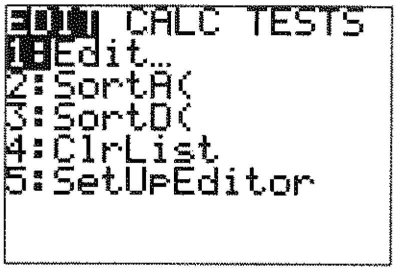
c. If the 2010 U.S. population was 309 million, which function better predicts the actual population?

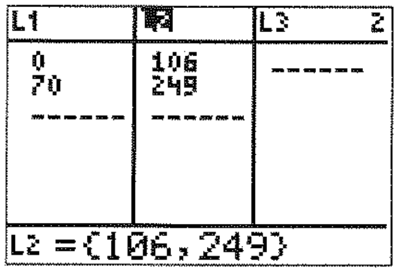
$f(90) = 1.27(90) + 106 = 220.3$ *and* $g(90) = 106(1.0123)^{90} \approx 318.5$. *The exponential model is closer.*

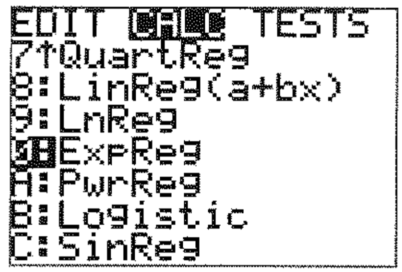
Finding the Exponential Regression (TI-84 Plus)

Directions for other calculators vary. Consult the owner's manual. Online graphing applications such as Desmos.com or a spreadsheet program can also find an exponential regression equation.

1. From your home screen, press STAT, and then from the STAT menu, select the EDIT option. (EDIT, ENTER)
2. Enter the x-values of the data set in L1, and enter the y-values of the data set in L2.
3. Select STAT. Move the cursor to the menu item CALC, and then move the cursor to option 0: ExpReg. Press ENTER.

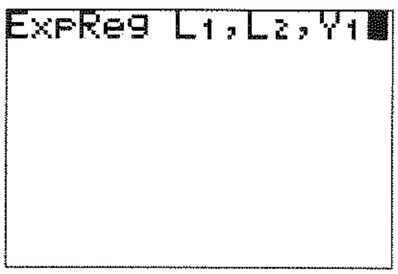


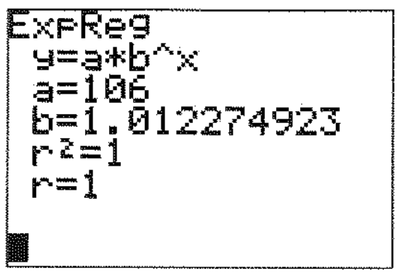


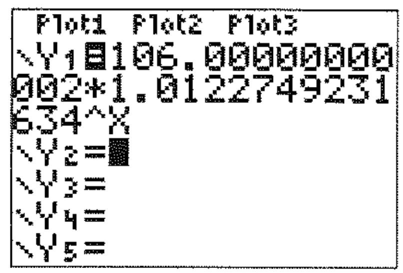


4. With ExpReg on the screen, enter L1, L2, and Y1 as described in the following notes.

   ExpReg L1, L2, Y1, and select ENTER to see results.







To obtain Y1, go to VARS, move the cursor to Y-VARS, and then FUNCTION (ENTER). You are now at the screen highlighting the y-variables. Move the cursor to Y1, and hit ENTER. Y1 is the exponential regression equation and will be stored in Y1.

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Lesson 15: Piecewise Functions

Graph a Step Function

1. The following piecewise function is an example of a step function.

$$G(x) = \begin{cases} 1, & 0 \le x < 1 \\ 2, & 1 \le x < 2 \\ 3, & 2 \le x < 3 \end{cases}$$

a. Graph this function, and state the domain and range.

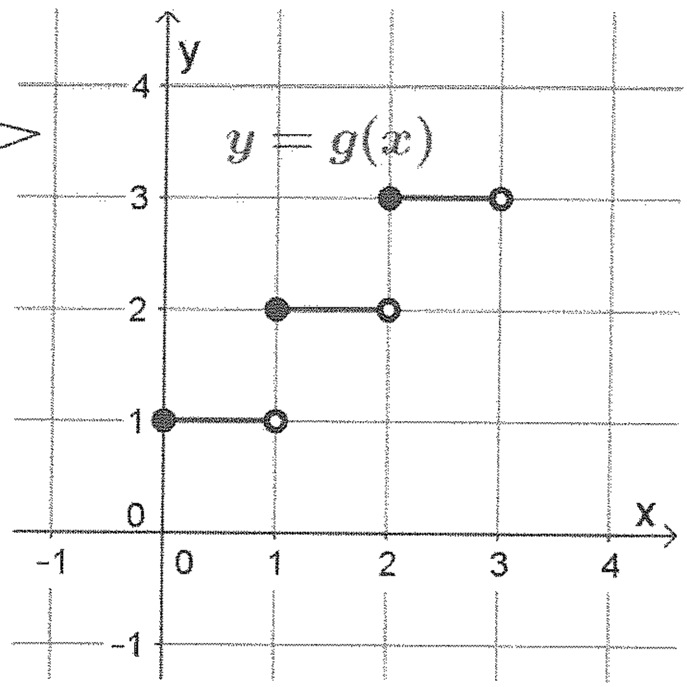
Domain: $[0, 3)$

Range: $\{1, 2, 3\}$

When writing the domain and range, I can use inequalities or interval notation. I learned how to write the domain and range in Module 3 Topic B.

I remember when looking at a graph of a step function:

- $<$ or $>$ (endpoint is OPEN) ○
- $\le$ or $\ge$ (endpoint is CLOSED) ●



b. Why is this type of function called a step function?

The graph resembles a set of stair steps.

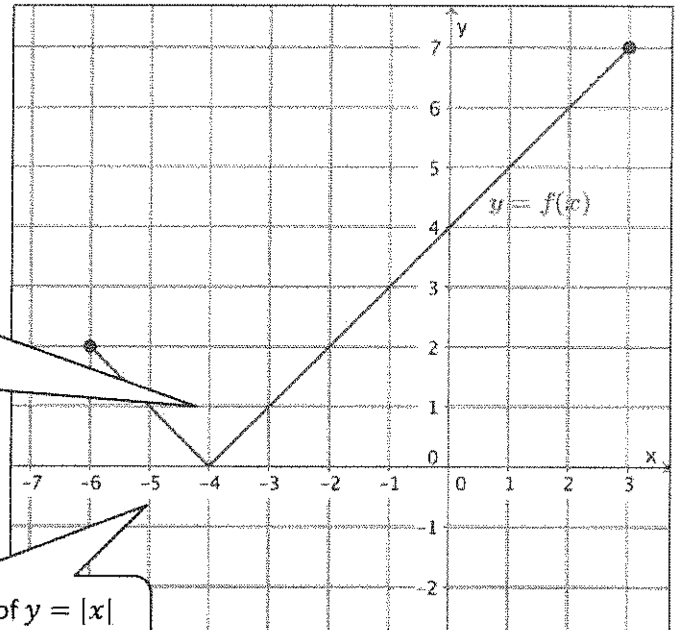
Graph Piecewise Functions Including Absolute Value Functions

2. Graph the following piecewise functions for the specified domain.

a. $f(x) = |x + 4|$ for $-6 \le x \le 3$

Graph $f(x) = -x - 4$ on the interval from -6 to -4.

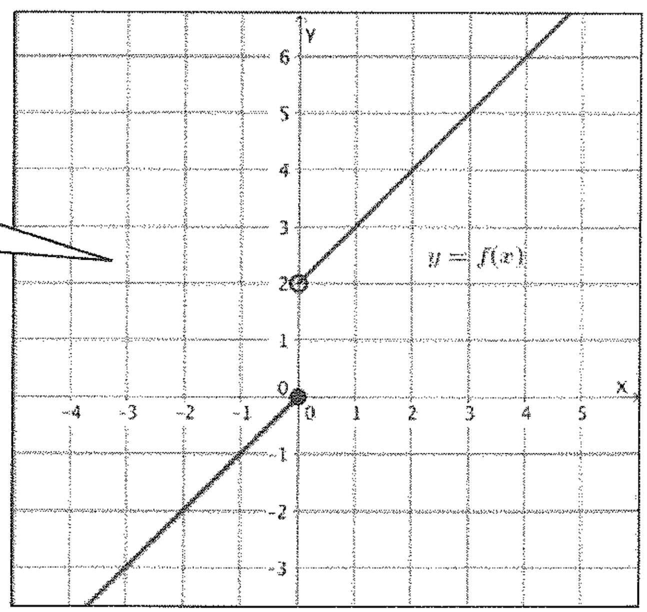
Graph $f(x) = x + 4$ on the interval from -4 to 3.



I rewrite the absolute value function as two linear equations in slope-intercept form. Then, I graph the piecewise function by connecting those two segments at their point of intersection.

This graph is the graph of $y = |x|$ translated 4 units to the left.

b. $f(x) = \begin{cases} x & \text{if } x \le 0 \\ x + 2 & \text{if } x > 0 \end{cases}$



I use the slope and y-intercept of each piece to graph the linear function on each interval.

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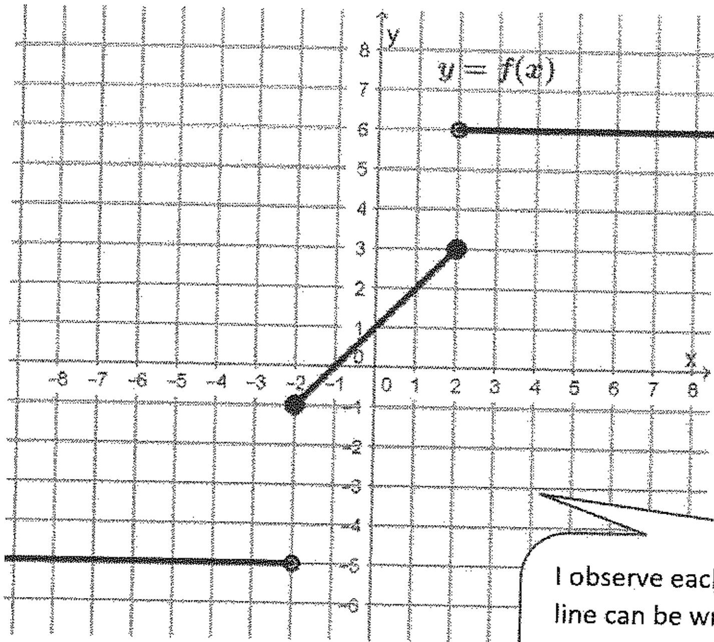
Write an Algebraic Function Given the Graph of the Function

3. Write a piecewise function for the graph below.



$$g(x) = \begin{cases} -5 & x < -2 \\ x + 1 & -2 \leq x \leq 2 \\ 6 & x > 2 \end{cases}$$

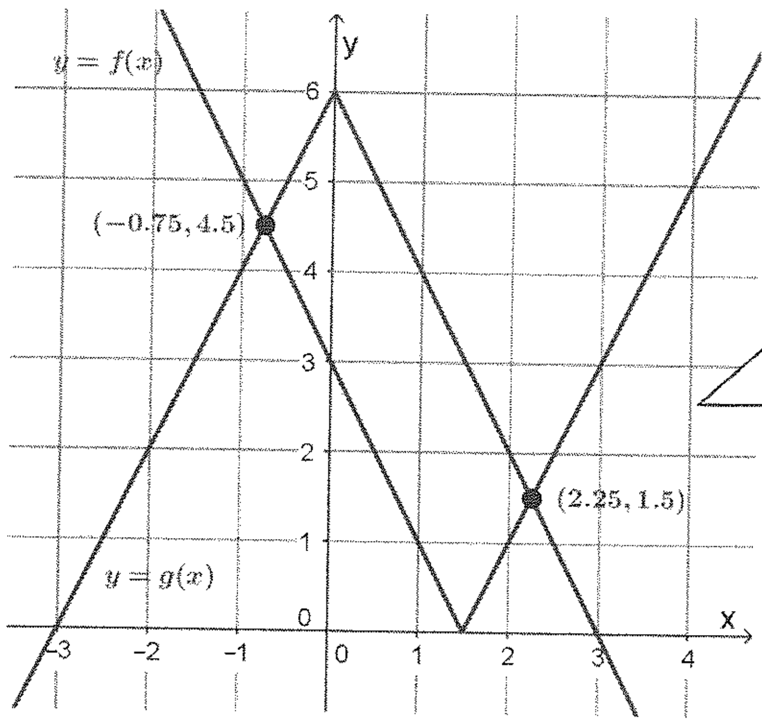
I observe each piece individually. The lower horizontal line can be written $y = -5$. I determine that the second equation is $y = x + 1$ by finding the slope and y-intercept. The third line is also horizontal, so its equation is $y = 6$. I use the endpoints to determine the domains. (See Problem 1.)

Lesson 16: Graphs Can Solve Equations Too

Solving an Equation Graphically

1. Solve the equation graphically: $|2x-3| = 6-|2x|$. Verify the solution set using the original equation.

Let** $f(x)=|2x-3|$ **and** $g(x)=6-|2x|$. **Graph each function and find the intersection points.



I enter both equations into the calculator equation editor. The graph shows the x-coordinates of intersection points, which are the solutions for the original equation.

Solutions: $\{-0.75, 2.25\}$

Verification: Substitute each solution into the original equation and see if it is a true number sentence.

Let $x=-0.75$***, then***

$$|2(-0.75)-3| = 6-|2(-0.75)|$$
$$4.5 = 4.5$$

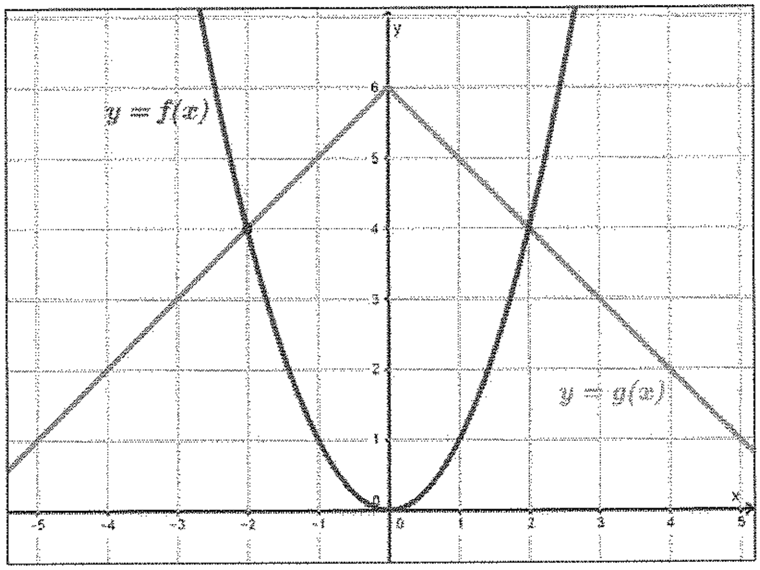
Let $x=2.25$***, then***

$$|2(2.25)-3| = 6-|2(2.25)|$$
$$1.5 = 1.5$$

Both sides the equation evaluate to the same number so the value of x is a solution.

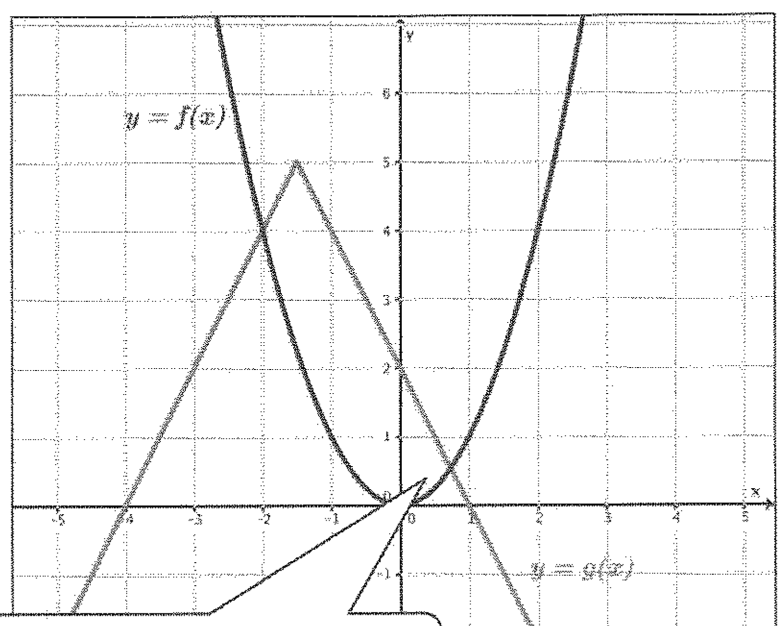
2. In the problem, the graphs of the functions f and g are shown on the same Cartesian plane. Estimate the solution set to the equation $f(x) = g(x)$. Assume that the graphs of the two functions intersect only at the points shown on the graph.

a.



The solution set appears to be $\{-2, 2\}$.

b.

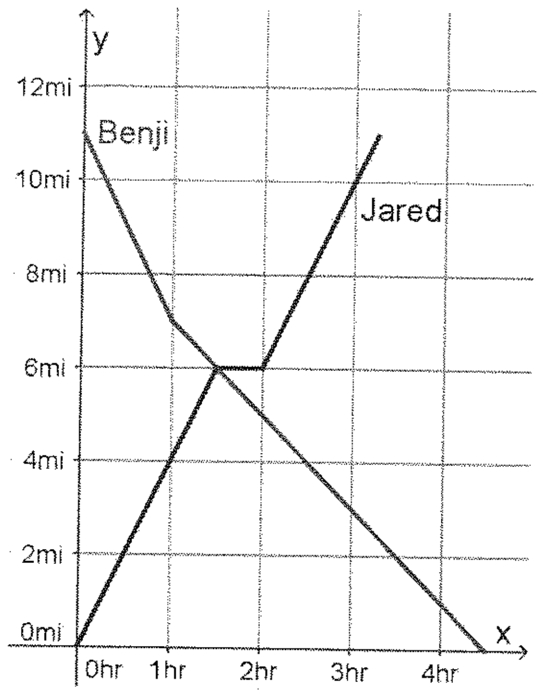


The solution set appears to be $\{-2, 0.75\}$.

I estimate the x-coordinate to be 0.75. It is between 0 and 1 but closer to 1.

Interpreting Intersection Points in a Real-World Situation

3. The graph below shows Jared's hike. He starts from one end of trail at the same time as Benji starts at the other end. Benji hikes at a steady pace of 4 miles per hour for the first hour and then slows down to 2 miles per hour for the rest of the hike. Jared hikes at steady pace of 4 miles per hour for the entire hike but stops and takes a 30-minute break after 1.5 hours.



a. Graph a function that represents Benji's distance from Jared's starting point as a function of time.

See the graph. Benji's graph is two pieces because he changes his speed after 1 hour.

b. Estimate when the two boys meet each other on the trail.

They cross paths at about 1.5 hours. I can tell that by estimating the x-coordinate of the intersection point of the graphs of the functions.

c. Write piecewise linear functions to represent each boy's distance, and use them to verify your answer to part (b).

Let $J(x)$ be Jared's hiking distance from his starting point after x hours.

$$J(x) = \begin{cases} 4x & 0 \le x \le 1.5 \\ 6 & 1.5 < x \le 2 \\ 4(x-2)+6 & 2 < x \le 3.25 \end{cases}$$

Let $B(x)$ be Benji's hiking distance from Jared's starting point after x hours.

$$B(x) = \begin{cases} 11 - 4x & 0 \le x \le 1 \\ -2(x-1)+7 & 1 < x \le 4.5 \end{cases}$$

Looking at the time interval when the graphs intersect, use Jared's first expression and Benji's second expression. Make an equation, and solve for x.

$$4x = -2(x-1)+7$$
$$4x = -2x+9$$
$$6x = 9$$
$$x = \frac{3}{2}$$

Their paths do cross at 1.5 hours.

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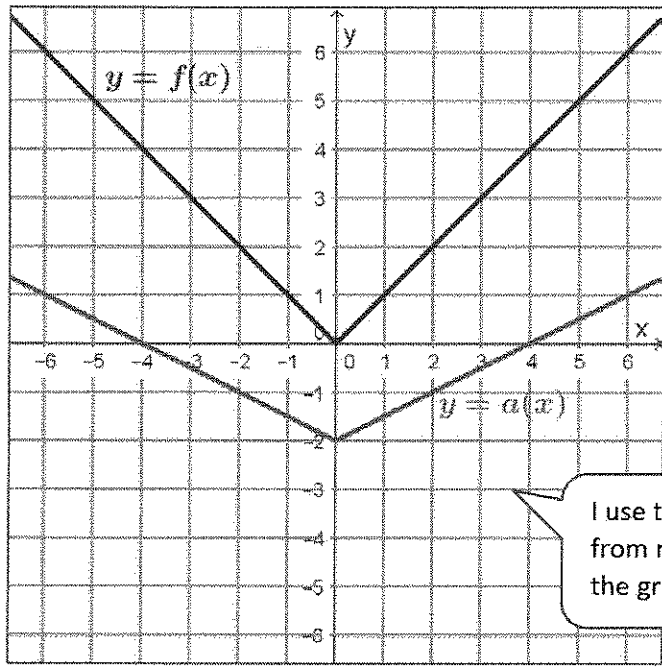
Lesson 17: Four Interesting Transformations of Functions

1. Let $f(x) = |x|$ for every real number x. The graph of $y = f(x)$ is shown below.



I use the ordered pairs from my table to create the graph of a.

a. Describe how the graph of the function below is a transformation of the graph of $y = f(x)$.

$$a(x) = \frac{1}{2}|x| - 2$$

Vertically scale/shrink the graph of $y = f(x)$ by dividing the output values by 2 for every input. Next, translate the graph of $y = f(x)$ down 2 units.

b. Complete the table below to generate output values for the function a.

x	$f(x) = \lvert x \rvert$	$a(x) = \frac{1}{2}\lvert x \rvert - 2$
-2	2	-1
-1	1	-1.5
0	0	-2
1	1	-1.5
2	2	-1

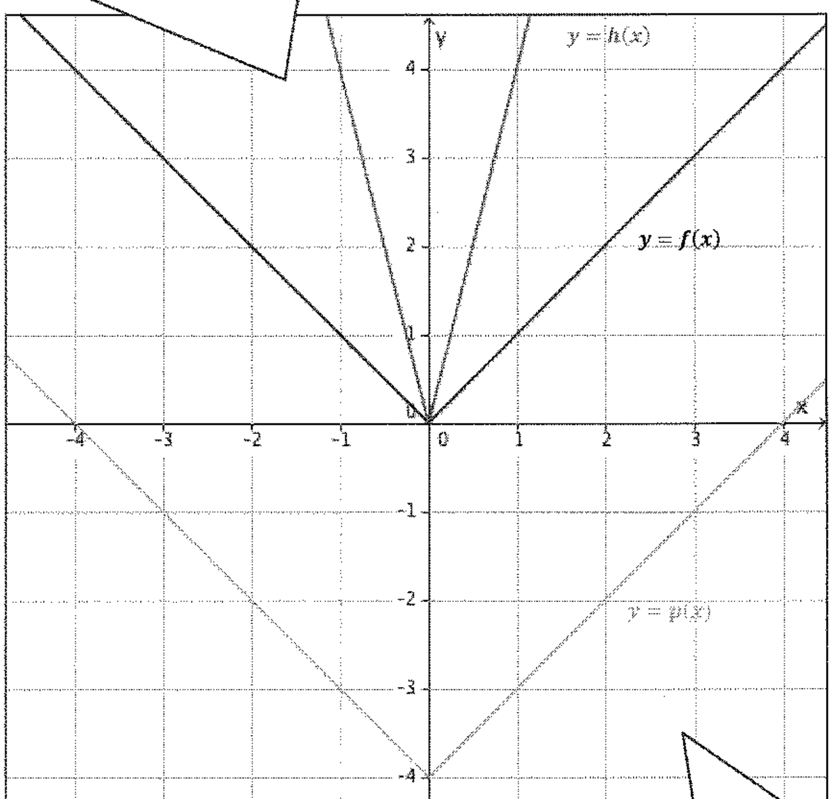
I can find the values of $a(x)$ in the table by taking each $f(x)$ value, multiplying it by $\frac{1}{2}$, and then subtracting by 2.

c. Use this same set of axes to graph the function. Be sure to label each function on your graph.

See the graph above.

2. Let $f(x) = |x|$ for every real number x. Let h and p be functions found by transforming the graph of $y = f(x)$. Use the graphs of $y = f(x)$, $y = h(x)$, and $y = p(x)$ below to write the functions h and p in terms of the function f.

For $y = h(x)$, it looks there is a vertical stretch/shrink and not translation. Comparing the point $(1, 1)$ on the graph of f to the point $(1, 4)$ on the graph of h, I can see there is a vertical stretch by a factor of 4.



In terms of $f(x)$,

$h(x) = 4f(x)$

$p(x) = f(x) - 4.$

For the graph of $y = p(x)$, I can see that if I translate the graph of f down 4 units, it will align with the graph of p, so there is no vertical stretch/shrink, only a vertical translation. I have to subtract 4 from $f(x)$ to obtain $p(x)$.

Lesson 18: Four Interesting Transformations of Functions

1. Let $f(x) = |x - 1|$ for every real number x. The graph of the equation $y = f(x)$ is provided on the Cartesian plane below. Transformations of the graph of $y = f(x)$ are described below. After each description, write the equation for the transformed graph. Then, sketch the graph of the equation you write for part (c).

   a. Scale the resulting graph vertically by a scale factor of 4.

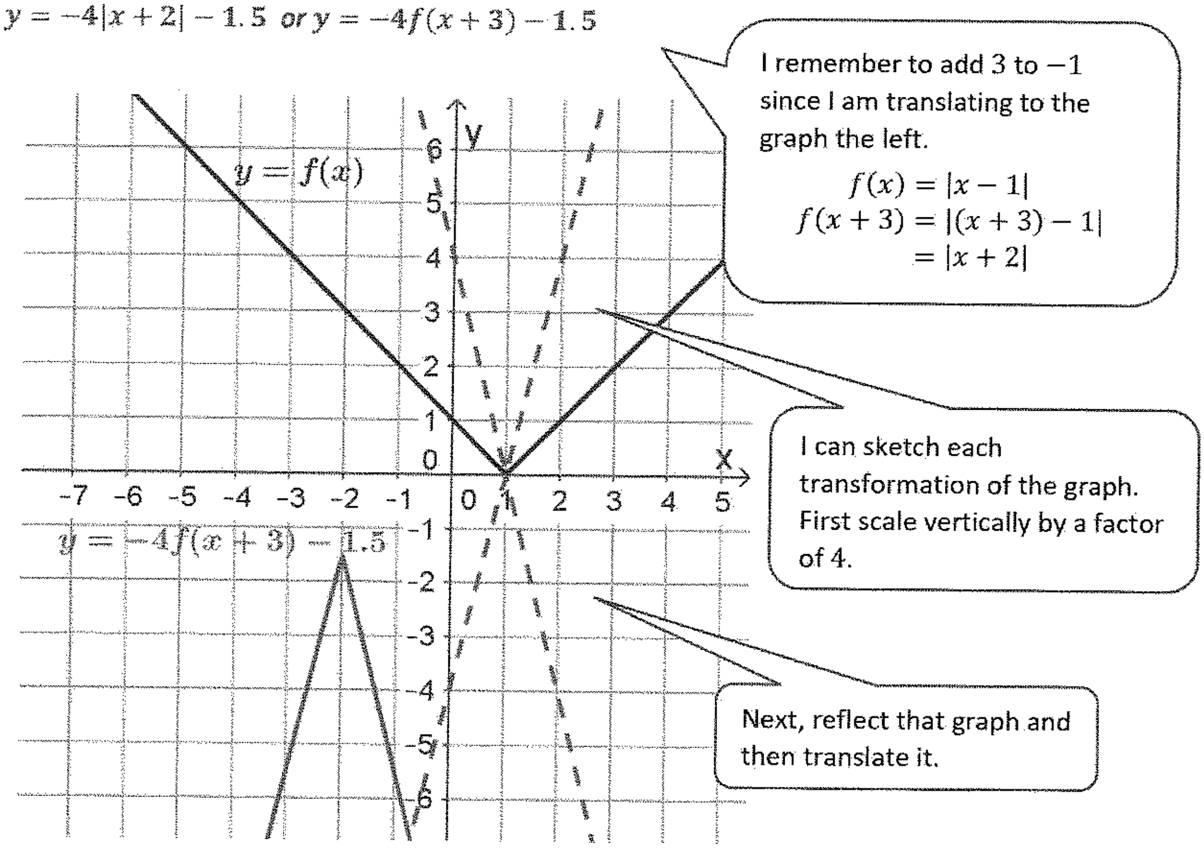
      $y = 4|x - 1|$ *or* $y = 4f(x)$

   b. Reflect the resulting graph from part (a) across the x-axis.

      $y = -4|x - 1|$ *or* $y = -4f(x)$

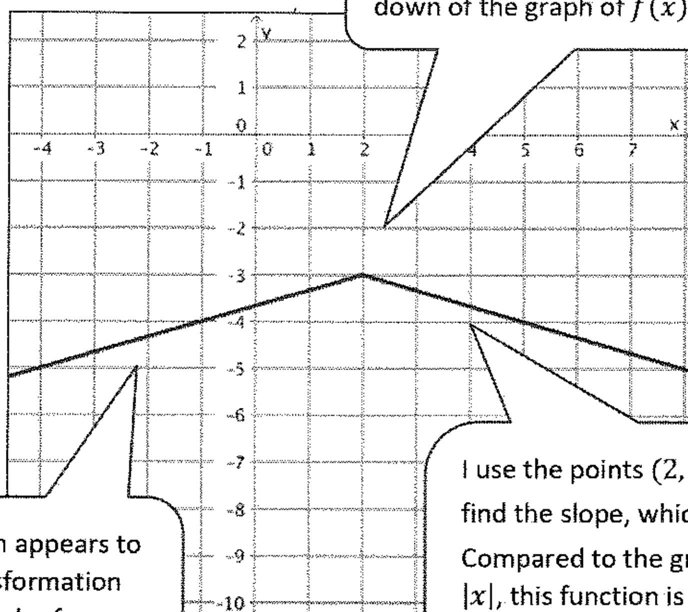
   c. Translate the resulting graph from part (b) left 3 units and down 1.5 units.

      $y = -4|x + 2| - 1.5$ *or* $y = -4f(x + 3) - 1.5$



2. Write the formula for the function depicted by the graph.

$b(x) = -\frac{1}{3}|x-2| - 3$



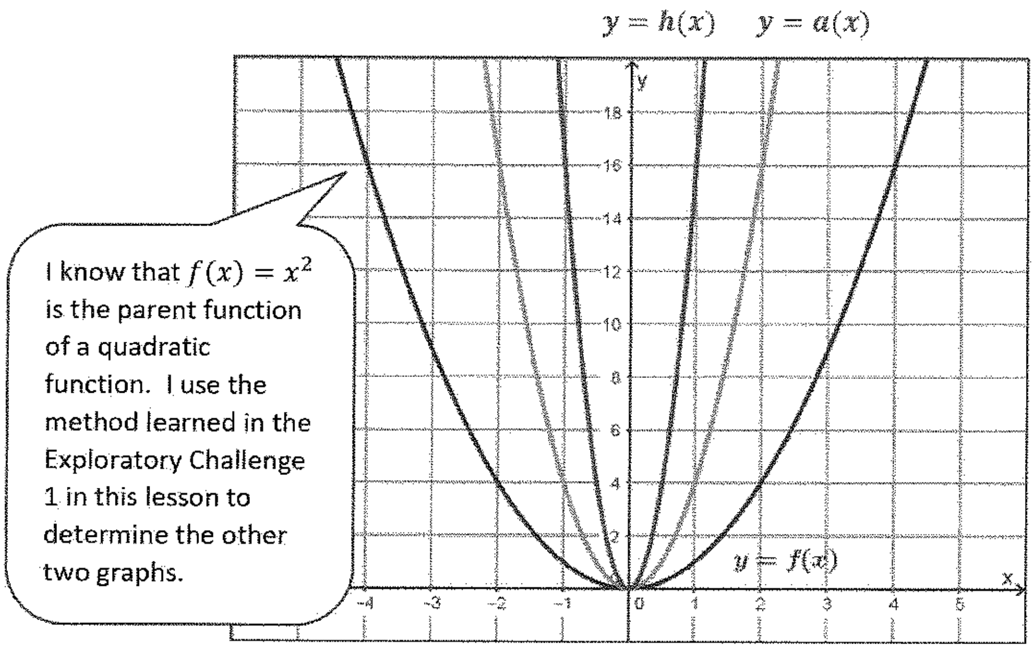
I see this point is $(2, -3)$, which indicates a shift of 2 units to the right and 3 units down of the graph of $f(x) = |x|$.

This graph appears to be a transformation of the graph of $f(x) = |x|$.

I use the points $(2, -3)$ to $(5, -4)$ to find the slope, which is $-\frac{1}{3}$. Compared to the graph of $f(x) = |x|$, this function is a vertical shrink by a factor of $\frac{1}{3}$ and a reflection across the horizontal axis.

Lesson 19: Four Interesting Transformations of Functions

Vertical and Horizontal Stretches and Shrinks



Let $f(x) = x^2$, $a(x) = 4x^2$, and $h(x) = (4x)^2$, where x can be any real number. The graphs above are of the functions $y = f(x)$, $y = a(x)$, and $y = h(x)$.

1. Label each graph with the appropriate equation.

   See the graph. The graph that includes the point $(1, 4)$ is the graph of a. The graph that includes the point $(1, 16)$ is the graph of h.

2. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = a(x)$. Use coordinates to illustrate an example of the correspondence.

The graph of $y = a(x)$ is a vertical stretch of the graph of $y = f(x)$ by scale factor 4; for a given x-value, the value of $a(x)$ is four times the value of $f(x)$. For example, the coordinate $(1, 1)$ is on the graph of f and the coordinate $(1, 4)$ is on the graph of a.

OR

The graph of $y = a(x)$ is a horizontal shrink of the graph of $y = f(x)$ by scale factor $\frac{1}{2}$. For any y-value, the x-value for function a is $\frac{1}{2}$ the x-value for function f. For example, the coordinate $(2, 4)$ is on the graph of f and the corresponding coordinate $(1, 4)$ is on the graph of a.

3. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = h(x)$. Use coordinates to illustrate an example of the correspondence.

The graph of $y = h(x)$ is a horizontal shrink of the graph of $y = f(x)$ by a scale factor of $\frac{1}{4}$. For any y-value, the x-value for function h is $\frac{1}{4}$ the x-value for function f. For example, the coordinate $(4, 16)$ is on the graph of f and the corresponding coordinate $(1, 16)$ is on the graph of h.

I remember that a horizontal scale of the graph of $y = f(x)$ with scale factor $k > 0$ is the graph of the equation $y = f\left(\frac{1}{k}x\right)$.

OR

The graph of $y = h(x)$ is a vertical stretch of the graph of $y = f(x)$ by scale factor 16; for a given x-value, the value of $h(x)$ is 16 times the value of $f(x)$. For example, the coordinate $(1, 1)$ is on the graph of f and the corresponding coordinate $(1, 16)$ is on the graph of h.

To see how these graphs can be both a vertical and horizontal scaling of the graph of f, I can rewrite the function a as a horizontal scaling and the function h as a vertical scaling.

$$a(x) = 4x^2 \text{ or } a(x) = (2x)^2$$
$$h(x) = (4x)^2 \text{ or } h(x) = 16x^2$$

Lesson 20: Four Interesting Transformations of Functions

Writing an Equation Given a Description of the Transformation

1. Suppose the graph of f is given. Write an equation for each of the following graphs after the graph of f has been transformed as described. Note that the transformations are not cumulative.

   a. Translate 2 units upward.

   $y = f(x) + 2$

   b. Translate 4 units downward.

   $y = f(x) - 4$

   c. Translate 8 units left.

   $y = f(x + 8)$

   > From the Opening Exercise in Lesson 20, I know this means k is -8, so I need to write $y = f(x - (-8))$, which is the same as $y = f(x + 8)$.

   d. Reflect about the x-axis.

   $y = -f(x)$

   e. Reflect about the y-axis.

   $y = f(-x)$

   > Multiplying $f(x)$ by -1 reflects the graph across the x-axis because the output values become their opposites. Multiplying x by -1 reflected the graph across the y-axis because the input values become their opposites.

   f. Shrink vertically by a factor of $\frac{1}{4}$.

   $y = \frac{1}{4}f(x)$

   g. Shrink horizontally by a factor of $\frac{1}{6}$.

   $y = f(6x)$

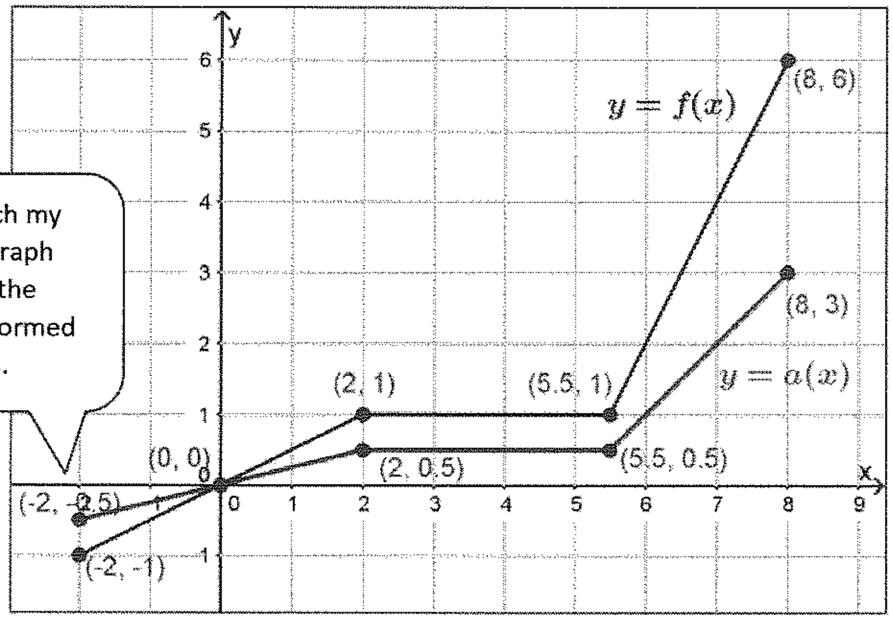
h. Stretch horizontally by a factor of 5.

$y = f\left(\frac{1}{5}x\right)$

Transformations of a Piecewise Linear Function

2. The graph $y = f(x)$ of a piecewise function f is shown. The domain of f is $-2 \leq x \leq 8$, and the range is $-1 \leq y \leq 6$.

I sketch my new graph using the transformed points.



a. Mark and identify at least five strategic points helpful in sketching the graph of $y = f(x)$.

$(-2, -1)$, $(0, 0)$, $(2, 1)$, $(5.5, 1)$, *and* $(8, 6)$

b. Use the graph to write the formula for the function f as a piecewise linear function.

$$f(x) = \begin{cases} 0.5x, & -2 \leq x \leq 2 \\ 1, & 2 < x < 5.5 \\ 2x - 10, & 5.5 \leq x \leq 8 \end{cases}$$

I find the equation of each segment using the strategic points. I recall my work from Lesson 15 to determine the domain for each line segment.

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c. Sketch the graph of $a(x) = \frac{1}{2}f(x)$, and state the domain and range of the transformed function. How can you use part (a) to help sketch the graph?

Domain: $-2 \le x \le 8$, range: $-0.5 \le y \le 3$. For every point (x, y) on the graph of $y = f(x)$, there is a point $\left(x, \frac{1}{2}y\right)$ on the graph of $y = \frac{1}{2}f(x)$. The strategic points on the graph of f can be used to determine the corresponding points on the graph of a. Use the same x-coordinate and half of the original y-coordinate: $(-2, -0.5)$, $(0, 0)$, $(2, 0.5)$, $(5.5, 0.5)$, and $(8, 3)$.

Lesson 21: Comparing Linear and Exponential Models Again

I recall that input refers to the domain and output refers to the range of a function.

Lesson Notes

Students increase their understanding of the fundamental differences between linear and exponential functions. Given a table of input-output pairs,

- If the *difference* between their corresponding outputs is always the same constant, then the input-output pairs in the table can be modeled by a *linear* function;
- If the *quotient* between their corresponding outputs is always the same constant, then the input-output pairs in the table can be modeled by an *exponential* function.

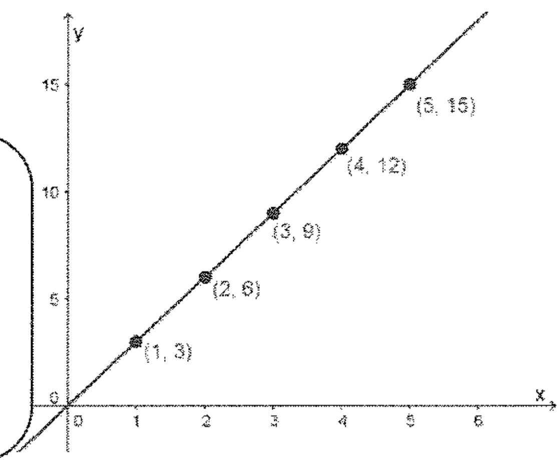
Identifying Data as Describing a Linear or an Exponential Relationship

For each table in Problems 1–4, classify the data as describing a linear relationship, an exponential growth relationship, an exponential decay relationship, or neither. If the relationship is linear, calculate the constant rate of change (slope), and write a formula for the linear function that models the data. If the function is exponential, calculate the common quotient for input values that are distance one apart, and write the formula for the exponential function that models the data. For each linear or exponential function found, graph the equation $y = f(x)$.

1.

x	$f(x)$
1	3
2	6
3	9
4	12
5	15

I need to check the table for a constant difference, and if that doesn't work, then check for a constant quotient. The input values increase by the same amount, so I can just subtract the output values.



$$f(2) - f(1) = f(3) - f(2) = f(4) - f(3) = f(5) - f(4) = 3$$

This input-output table has a constant rate of change of 3, so the data represent a linear relationship.

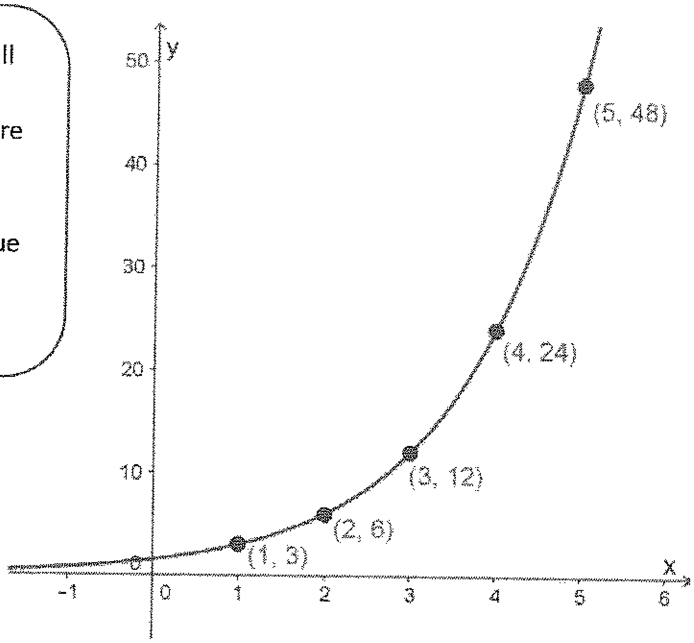
$f(x) = 3x + 0$

When the input values increase by 1, the difference of the output values is the constant rate of change or slope. When I graph the ordered pairs $(x, f(x))$ from the table, I can see the y-intercept is 0.

2.

x	$f(x)$
1	3
2	6
3	12
4	24
5	48

I don't need to check all the differences when I see that the first two are not the same. To find the quotient, I need to divide each output value in the table by the previous one.



$f(2) - f(1) \neq f(3) - f(2)$

This input-output table does not represent a linear relationship.

Check for a common quotient:

$$\frac{f(2)}{f(1)} = \frac{f(3)}{f(2)} = \frac{f(4)}{f(3)} = \frac{f(5)}{f(4)} = 2$$

The quotients are all equal to 2. This input-output table represents an exponential growth relationship with a common quotient of 2.

$f(x) = 3(2)^{x-1}$

I can use the formula for a geometric sequence to help me write the exponential function. I need to adjust the exponent so $f(1) = 3$.

3.

x	$f(x)$
1	3
2	6
3	11
4	18
5	27

I check for a linear function by quickly subtracting consecutive output values or an exponential function by quickly dividing consecutive output values. I am looking for constant differences or quotients.

$6 - 3 = 3$ and $11 - 6 = 5$

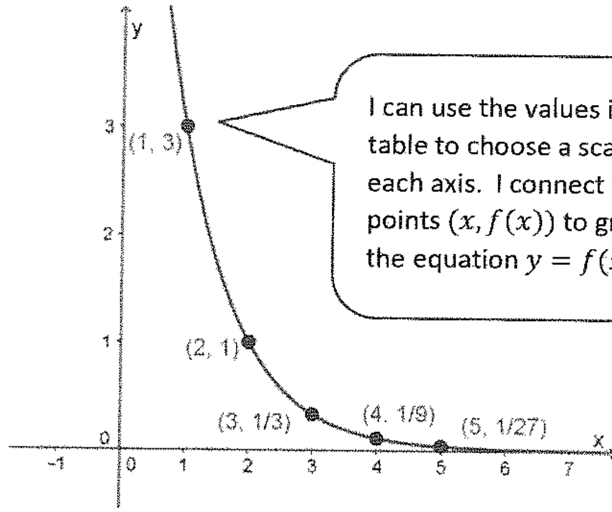
$\frac{6}{3} = 2$ and $\frac{11}{6} = 1.8\overline{3}$

$f(2) - f(1) \neq f(3) - f(2)$ ***and*** $\frac{f(2)}{f(1)} \neq \frac{f(3)}{f(2)}$

This input-output table does not represent a linear or an exponential relationship.

4.

x	$f(x)$
1	3
2	1
3	$\frac{1}{3}$
4	$\frac{1}{9}$
5	$\frac{1}{27}$



I can use the values in the table to choose a scale for each axis. I connect the points $(x, f(x))$ to graph the equation $y = f(x)$.

$f(2) - f(1) \neq f(3) - f(2)$

This input-ouput table does not represent a linear relationship.

Check for a common quotient.

$$\frac{f(2)}{f(1)} = \frac{f(3)}{f(2)} = \frac{f(4)}{f(3)} = \frac{f(5)}{f(4)} = \frac{1}{3}$$

This input-output table has common quotient of $\frac{1}{3}$***. It represents an exponential decay relationship.***

$$f(x) = 3\left(\frac{1}{3}\right)^{x-1}$$

An exponential function has the form $f(x) = a \cdot b^x$. The variable is in the exponent.

Applications of Exponential and Linear Functions

5. The population in a small town is modeled by the function $P(t) = 2000(1.05)^t$, where t is the years since 2010.

   a. Explain, in terms of the structure of the expression used to define $P(t)$, why the population after the year 2010 will never be 2,000?

   The function P is an example of exponential growth because the base is 1.05***, which is greater than*** 1***. The domain that makes sense in this situation is*** $t \geq 0$ ***because we are tracking the growth since the year 2010.*** $P(0) = 2000(1.05)^0 = 2000$***. The values of*** $P(t)$ ***will be greater than*** 2000 ***for all*** $t > 0$ ***because the function is increasing.***

Exponential growth functions are always increasing. Exponential decay functions are always decreasing.

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b. By what percent does $P(t)$ grow each year? Explain by writing a recursive formula for the sequence $P(0), P(1), P(2)$, etc.

This means I need to write a formula for $P(t+1)$ in terms of $P(t)$.

Writing out the sequence, we see

I can look for patterns to help me write the formula.

$$P(0) = 2000$$
$$P(1) = 2100 = 2000 + 0.05(2000)$$
$$P(2) = 2205 = 2100 + 0.05(2100)$$
$$P(3) = 2315.25 = 2205 + 0.05(2205)$$

Thus,

$$\begin{aligned} P(t+1) &= P(t) + 0.05P(t) \\ &= P(t)(1 + 0.05) \\ &= 1.05 \cdot P(t) \end{aligned}$$

I know the 1 represents one whole or 100% and the 0.05 represents a 5% increase.

This shows that $P(t)$ grows by 5% per year.

c. By what percentage does the population grow every 5 years?

$$\begin{aligned} P(t+5) &= P(t+4) \cdot 1.05 \\ &= P(t+3) \cdot 1.05 \cdot 1.05 \\ &= P(t+2) \cdot 1.05 \cdot 1.05 \cdot 1.05 \\ &= P(t+1) \cdot 1.05 \cdot 1.05 \cdot 1.05 \cdot 1.05 \\ &= P(t) \cdot 1.05 \cdot 1.05 \cdot 1.05 \cdot 1.05 \cdot 1.05 \\ &= P(t) \cdot (1.05)^5 \end{aligned}$$

$1.05^5 \approx 1.28$*, so the population grows by approximately 28% every 5 years.*

d. Write a linear function to model the population growth so that the linear function overestimates the population for the first 5 years after 2010 and underestimates it after the year 2015 when compared to the exponential function P.

The population in 2010 is $2,000$. $P(5) = 2553$ and $P(6) = 2680$ rounded to the nearest whole number. The linear function $f(t) = mt + 2000$ must be larger than $2,553$ when $t = 5$ and smaller than $2,680$ when $t = 6$.

Solve the compound inequality:

$2553 < 5m + 2000$ *and* $2680 > 6m + 2000$

$m > 110.6$ *and* $m < 113.\overline{3}$

I need compare my linear function values to the values of P when $t = 5$ and $t = 6$.

Choose any real number between these values to find a linear function that meets the specified conditions. One possible function is $f(t) = 112t + 2000$.

Lesson 22: Modeling an Invasive Species Population

Lesson Notes

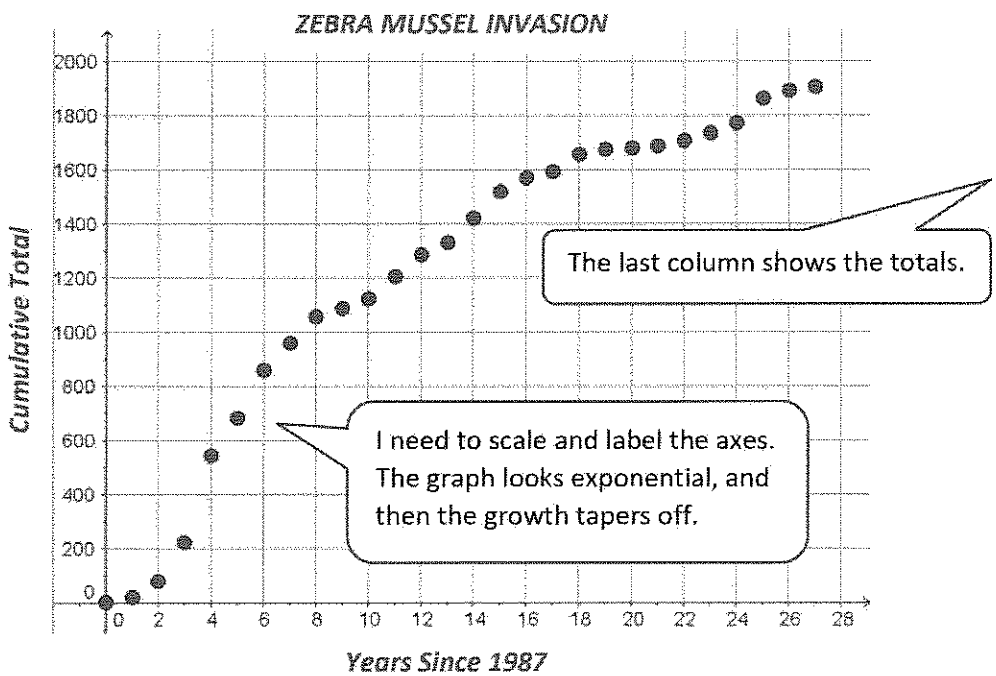
This lesson and the problem set explore invasive species. The problem set asks students to research an invasive plant and report on it. The report should include data tables or graphs that detail the spread.

Modeling Invasive Species

Since 1988, the zebra mussel has invaded the Great Lakes region of the United States and caused damage to the ecosystem. The data to the right provide a count of zebra mussels found in Great Lakes drainages. (USGS. 2015. Nonindigenous Aquatic Species Database, Gainesville, FL. http://nas.er.usgs.gov).

Year	Count	Total
1987	1	1
1988	19	20
1989	57	77
1990	146	223
1991	319	542
1992	140	682
1993	175	857
1994	100	957
1995	98	1055
1996	30	1085
1997	37	1122
1998	81	1203
1999	80	1283
2000	47	1330
2001	88	1418
2002	98	1516
2003	51	1567
2004	25	1592
2005	64	1656
2006	18	1674
2007	5	1679
2008	7	1686
2009	21	1707
2010	28	1735
2011	35	1770
2012	93	1863
2013	28	1891
2014	13	1904

1. Create a table and graph for the cumulative total number of zebra mussels.



2. Would an exponential or linear function better model this data? Explain your reasoning.

   An exponential function would be a good model for the first 6 to 8 years. After that, the data appears to be more linear.

   I could use a graphing calculator to create a scatter plot and find a regression equation for each part.

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Lesson 23: Newton's Law of Cooling

Lesson Notes

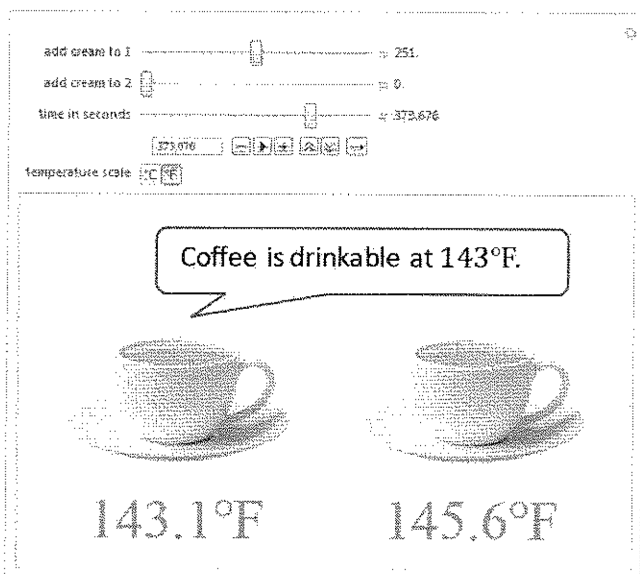
Students explore transformations of exponential functions and apply their knowledge to a famous relationship known as Newton's Law of Cooling. The Wolfram Demonstrations application is used to explore this formula in the Problem Set.

The Coffee Cooling Problem Application

Use the Wolfram Demonstration referenced in the Problem Set to answer these questions.

> The Problem Set directions provide a web address and directions for downloading the viewer I need to answer the questions on the problem set. I can adjust the sliders to answer the questions. I can slow down or speed up the animation as well.

The Coffee Cooling Problem



1. With no cream added, what is the temperature of the coffee after 500 seconds?

   Set slider 1 to the far right, and run the animation. The temperature is $152.4°F$.

2. How long does it take a cup of coffee to become drinkable if I add cream at the beginning?

   Set slider 1 to the far left, and run the animation. The temperature is about $143°F$ after 428 seconds.

3. How long does it take the coffee to become drinkable if I add cream around 250 seconds?

   Set slider 1 to 250 seconds, and run the animation. It takes about 375 seconds.

4. Which cup will be drinkable sooner, one with cream added at the beginning or one with cream added after 50 seconds?

   Set slider 1 to the far left and slider 2 to 50 seconds, and run the animation. Both cups are drinkable around 415 seconds, but Cup 2 (with cream added later) is drinkable just a bit sooner.

Lesson 24: Piecewise and Step Functions in Context

Write a Piecewise Linear Function from a Verbal Description of a Situation

The table below provides information about the 2014 federal income tax brackets for single filers.

Rate	Single Filers
10%	\$0 to \$9,075
15%	\$9,076 to \$36,900
25%	\$36,901 to \$89,350
28%	\$89,351 to \$186,350
33%	\$186,351 to \$405,100
35%	\$405,101 to \$406,750
39.6%	\$406,751 +

There will be 7 pieces for the function. I can model each piece with linear function because the tax rate is constant on each interval.

1. Write a piecewise function that gives the tax owed based on this table for incomes from \$0 to \$500,000.

$$f(x)=\begin{cases} 0.1x, & 0\le x<9076 \\ 0.15(x-9076)+907.6, & 9076\le x<36901 \\ 0.25(x-36901)+5081.35, & 36901\le x<89351 \\ 0.28(x-89351)+18193.85, & 89351\le x<186351 \\ 0.33(x-186351)+45353.85, & 186351\le x<405101 \\ 0.35(x-405101)+117541.35, & 405101\le x<406751 \\ 0.396(x-406751)+118118.85, & 406751\le x\le 500000 \end{cases}$$

The tax rate is the slope of each linear function. To write a linear expression for each piece, find the output value of each piece at the end of the interval, and use that in the next piece.

For example, when $x=9076$, the output value in the first expression is $0.1(9076)=907.6$. The second expression will have a slope equal to the decimal value of the second tax rate. The starting point of the linear function needs to be shifted to the ending point of the previous part. So, the second expression must represent a translation right 9076 units and up 907.6 units.

Similarly each subsequent piece will have a slope equal to the tax rate in decimal form and then be translated horizontally and vertically. The 5081.35 in the third expression comes from evaluating the second expression at the right endpoint.

$$0.15(36901-9076)+907.6=5081.35$$

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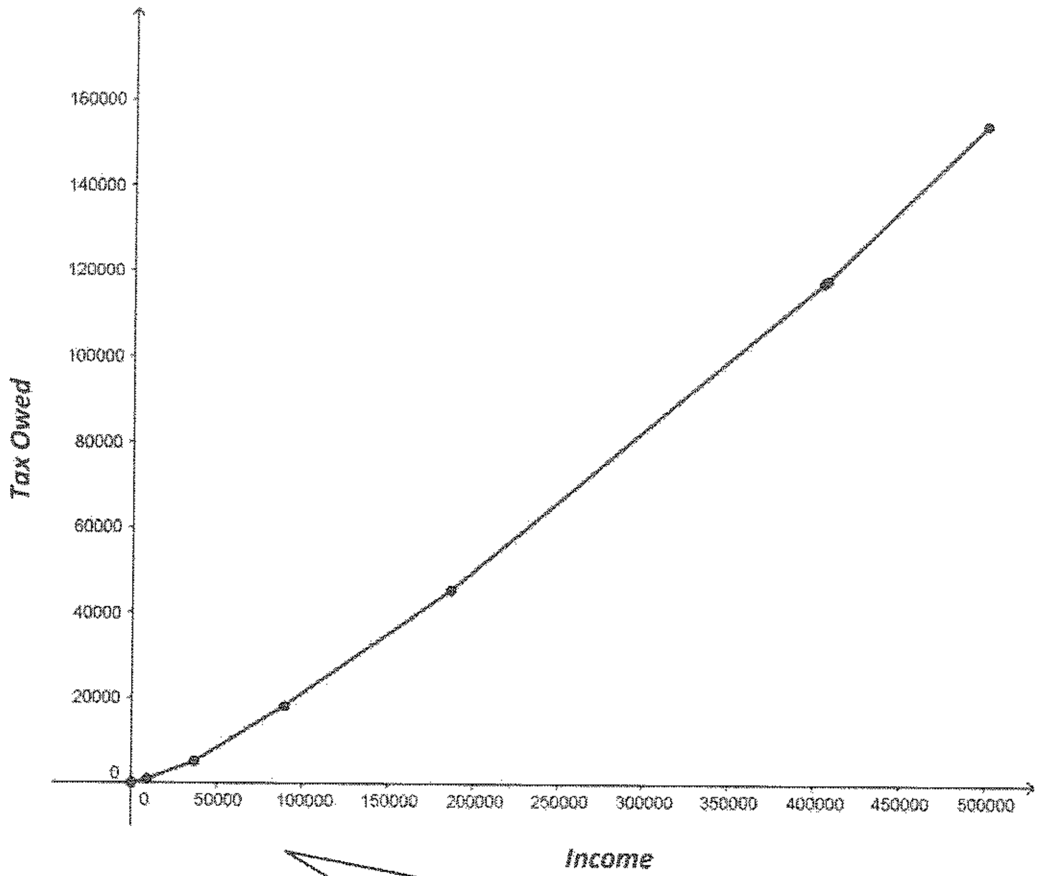
2. Graph the function you wrote in Problem 1.



The graph is composed of 7 different straight line segments. I can use the function to generate a set of points $(x, f(x))$ and then graph them and connect them with straight line segments.

3. Use your function to calculate the tax owed if you earned $75,000 in 2014.

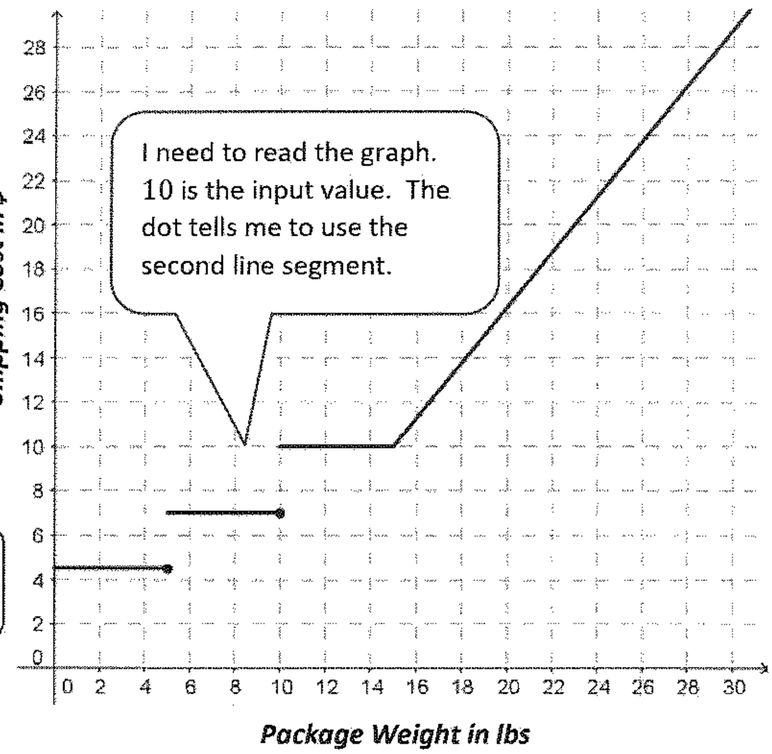
Use the third expression. Substituting $75,000$ *for x gives*

$0.25(75000 - 36901) + 5081.35 = 14606.1.$

You will owe $\$14,606.10.$

Use a Step Function to Answer Questions Related to a Situation

The graph shows shipping charges for different package weights.



4. How much would it cost to ship a package that weighs 10 lbs.?

   To ship a 10 lb. package, it would cost you $\$7$ shipping.

5. How much would it cost to ship a package that weighs 20 lbs.?

   To ship a 20 lb. package, it would cost you about $\$16$.

   I can only estimate this unless I have the function given algebraically.

Transformations of the Graphs of Step Functions

6. Describe the graph of g, shown to the right, as a transformation of the graph of the floor function, $f(x) = \lfloor x \rfloor$.

   The graph has been scaled vertically by a factor of two and shifted one unit to the right.

   Thus, $g(x) = 2 \cdot f(x - 1)$

   I learned about transformations of the graph of a function in Topic C.