Lesson 1: Integer Sequences—Should You Believe in Patterns?

Generating Terms of a Sequence

1. Consider a sequence given by the formula $f(n) = 12 - 7n$ starting with $n = 1$. Generate the first 5 terms of the sequence.

$f(1) = 12 - 7(1) = 5$
$f(2) = 12 - 7(2) = -2$
$f(3) = 12 - 7(3) = -9$
$f(4) = 12 - 7(4) = -16$
$f(5) = 12 - 7(5) = -23$

I see that this sequence has a "minus 7" pattern. I could use this pattern to continue generating terms in the sequence.

To find the first five terms of the sequence, I can replace n with the numbers 1, 2, 3, 4, and 5.

The first five terms of the sequence are $5, -2, -9, -16, -23$.

Writing a Formula for a Sequence

2. Consider the following sequence that follows a times $\frac{1}{2}$ pattern: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

   a. Write a formula for the n^{th} term of the sequence. Be sure to specify what value of n your formula starts with.

   $f(n) = \left(\frac{1}{2}\right)^{n-1}$ *starting with* $n = 1$

   I know that my formula can start with any value of n but that the convention is to start with $n = 1$.

   I can check my formula by using it to generate the first few terms of the sequence.

   $f(1) = \left(\frac{1}{2}\right)^{1-1} = \left(\frac{1}{2}\right)^0 = 1$

   $f(2) = \left(\frac{1}{2}\right)^{2-1} = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$

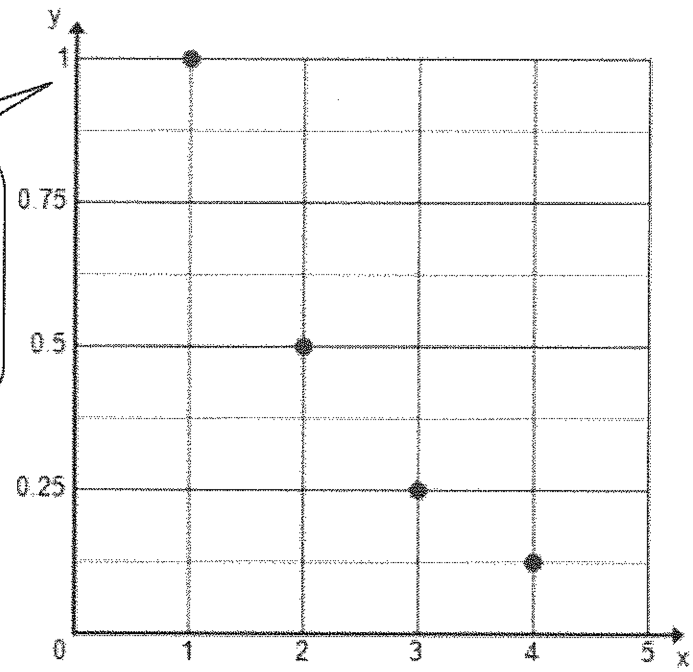
b. Using the formula, find the 10^{th} term of the sequence.

$$f(10) = \left(\frac{1}{2}\right)^{10-1} = \left(\frac{1}{2}\right)^{9} = \frac{1}{512}$$

Since my formula starts with $n = 1$, I can find the 10^{th} term by replacing n with 10.

c. Graph the four terms of the sequence as ordered pairs $(n, f(n))$ on a coordinate plane.

The ordered pair for the first term is $(1, 1)$. The ordered pair for the second term is $\left(2, \frac{1}{2}\right)$, and so on.



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Lesson 2: Recursive Formulas for Sequences

Generating Terms of a Sequence When Given a Recursive Formula

List the first five terms of each sequence.

1. $f(n) = -3f(n-1)$ and $f(1) = 2$ for $n \geq 2$

> I can find the next term in the sequence by starting with $n = 2$.

> I see that this is a recursive formula. To find any term in the sequence, I need to use the previous term.

> This means that the first term of the sequence is 2. I can use that to get started.

> I replaced n with 2 in the formula to find the second term.

$f(1) = 2$

$f(2) = -3f(2-1) = -3f(1) = -3(2) = -6$

$f(3) = -3f(3-1) = -3f(2) = -3(-6) = 18$

$f(4) = -3f(4-1) = -3f(3) = -3(18) = -54$

$f(5) = -3f(5-1) = -3f(4) = -3(-54) = 162$

The first five terms of the sequence are $2, -6, 18, -54, 162$.

2. $a_{n+1} = a_n + 2n + 1$ where $a_1 = 1$ for $n \geq 1$

> This sequence is valid for integers greater than or equal to 1.

> The subscripts in this notation represent the term number just like the values in the parentheses did in the formula above.

> I replaced n with 1 in the formula to find the second term.

$a_1 = 1$

$a_2 = a_1 + 2(1) + 1 = 1 + 2 + 1 = 4$

$a_3 = a_2 + 2(2) + 1 = 4 + 4 + 1 = 9$

$a_4 = a_3 + 2(3) + 1 = 9 + 6 + 1 = 16$

$a_5 = a_4 + 2(4) + 1 = 16 + 8 + 1 = 25$

The first five terms of the sequence are $1, 4, 9, 16, 25$.

Writing a Recursive Formula for a Sequence

3. Write a recursive formula for the sequence that has an explicit formula $f(n) = 4n - 2$ for $n \geq 1$.

$f(1) = 4(1) - 2 = 2$

$f(2) = 4(2) - 2 = 6$

$f(3) = 4(3) - 2 = 10$

$f(4) = 4(4) - 2 = 14$

I see that this sequence is following a "plus 4" pattern.

It might be helpful to generate the first few terms of the sequence.

$f(n+1) = f(n) + 4$ *where* $f(1) = 2$ *and* $n \geq 1$

4. The bacteria culture has an initial population of 100 and it quadruples in size every hour.

This sequence has a "times 4" pattern: $100, 400, 1600, 6400$

$B_{n+1} = 4B_n$ *where* $B_1 = 100$ *and* $n \geq 1$

I can use subscripts or parentheses like $B(n+1)$ to name the sequence.

Each term in the sequence is 4 times the previous one.

$400 = 4 \cdot 100$

$1600 = 4 \cdot 400$

$6400 = 4 \cdot 1600$

Noticing this pattern helps me write the recursive formula.

Lesson 3: Arithmetic and Geometric Sequences

Identifying Sequences as Arithmetic or Geometric

1. List the first five terms of the sequence given below, and identify it as arithmetic or geometric.

$$A(n+1) = -3 \cdot A(n) \text{ for } n \geq 1 \text{ and } A(1) = 2$$

$A(1) = 2$

$A(2) = -3 \cdot A(1) = -3 \cdot 2 = -6$

$A(3) = -3 \cdot A(2) = -3 \cdot -6 = 18$

$A(4) = -3 \cdot A(3) = -3 \cdot 18 = -54$

$A(5) = -3 \cdot A(4) = -3 \cdot -54 = 162$

> I was given a recursive formula and the first term, $A(1)$. I can use the first term to find the second term.

This sequence is geometric.

> I see that each term in the sequence is the product of the previous term and -3.

2. Identify the sequence as arithmetic or geometric, and write a recursive formula for the sequence. Be sure to identify your starting value.

$$15, 9, 3, -3, -9, \ldots$$

This sequence is arithmetic.

> I see that each term in the sequence is the sum of the previous term and -6.

$f(n+1) = f(n) - 6$ *for* $n \geq 1$ *and* $f(1) = 15$

Writing the Explicit Form of an Arithmetic or Geometric Sequence

3. Consider the arithmetic sequence $15, 9, 3, -3, -9, \ldots$.

   a. Find an explicit form for the sequence in terms of n.

   $f(n) = 15 + (n-1) \cdot -6 = -6n + 21$ *for* $n \geq 1$

> I need to identify the pattern. To find the second term, I need to subtract 6 one time. To find the third term, I need to subtract 6 two times. To find the n^{th} term, I need to subtract 6 $(n-1)$ times.

b. Find the 80^{th} term.

$$f(80) = -6(80) + 21 = -459$$

To find the 80^{th} term, I need to find $f(80)$ using the explicit form.

4. Find the common ratio and an explicit form for the following geometric sequence.

$$108, 36, 12, 4, \ldots$$

I can find the common ratio, r, by dividing any two successive terms.

$$r = \frac{36}{108} = \frac{12}{36} = \frac{4}{12} = \frac{1}{3}$$

I need to identify the pattern. To find the second term, I need to multiply by $\frac{1}{3}$ one time. To find the third term, I need to multiply by $\frac{1}{3}$ two times. To find the n^{th} term, I need to multiply by $\frac{1}{3}$ $(n-1)$

$$f(n) = 108\left(\frac{1}{3}\right)^{n-1}$$

I can check my formula by finding a term in the sequence.

$f(4) = 108\left(\frac{1}{3}\right)^{4-1} = 108\left(\frac{1}{3}\right)^{3} = 4$

The fourth term of the sequence is 4.

5. The first term in an arithmetic sequence is 2, and the 5^{th} term is 8. Find an explicit form for the arithmetic sequence.

$$8 = 2 + 4 \cdot d$$

$$\frac{3}{2} = d$$

2, , , ,8

I need to find some number, d, that when added to the first term 4 times results in the fifth term.

$$f(n) = 2 + (n-1)\cdot\frac{3}{2} = \frac{3}{2}n + \frac{1}{2}$$

I can check my formula by finding a term in the sequence.

$f(5) = \frac{3}{2}(5) + \frac{1}{2} = 8$

The fifth term of the sequence is 8.

Lesson 4: Why Do Banks Pay YOU to Provide Their Services?

Calculations Involving Simple Interest

1. \$800 is invested at a bank that pays 5% simple interest. Calculate the amount of money in the account after 12 years.

$I(t) = Prt$

> I can use this formula to calculate the interest. P is the principal amount, and r is the interest rate in decimal form.

> I know that simple interest means that interest is earned only on the original investment amount.

$I(12) = 800(0.05)(12)$

$I(12) = 480$

After 12 *years, the account will have* $\$1,280$.

Calculations Involving Compound Interest

2. \$800 is invested at a bank that pays 5% interest compounded annually. Calculate the amount of money in the account after 12 years.

$FV = P(1+r)^n$

> I can use this formula to calculate the future value n years after investing.

> I know that compound interest means that each time interest is earned, it becomes part of the principal.

$FV = 800(1+0.05)^{12}$

$\approx 1,436.69$

After 12 *years, the account will have* $\$1,436.69$.

Lesson 5: The Power of Exponential Growth

1. In the year 2000, a total of 768,586 high school students took an Advanced Placement (AP) exam. Since the year 2000, the number of high school students who take an AP exam has increased at an approximate rate of 9% per year.

   a. What explicit formula models this situation?

   $f(t) = 768586(1.09)^t$,

   where t represents the number of years since 2000.

   > This is an example of exponential growth, so I need an explicit formula for a geometric sequence to model this situation.

   > In this formula, I am starting with $t = 0$ (the year 2000).

   b. If this trend continues, predict the number of high school students who will take an AP exam in the year 2020.

   $f(20) = 768586(1.09)^{20} = 4307471.654$

   > Since t represents years since 2000, I need to evaluate $f(20)$.

   If this trend continues, approximately 4,307,472 students will take an AP exam in the year 2020.

2. Jackie decided to start a savings plan where she deposited \$0.01 in a jar on day one, \$0.03 on day two, \$0.09 on day three, and so on, tripling the amount she saved each day. After how many days of following this plan would the amount she deposited in the jar exceed \$10,000? Be sure to include an explicit formula with your response.

   > I see that the amount she saves each day forms a geometric sequence with a "times 3" pattern.

   $A(n) = 0.01(3)^{n-1}$ *for* $n \geq 1$

   > I can check my formula.
   > Day 1: $A(1) = 0.01(3)^{1-1} = 0.01$
   > Day 2: $A(2) = 0.01(3)^{2-1} = 0.03$
   > Day 3: $A(3) = 0.01(3)^{3-1} = 0.09$

   $A(13) = 0.01(3)^{13-1} = 0.01(3)^{12} = 5314.41$

   $A(14) = 0.01(3)^{14-1} = 0.01(3)^{13} = 15943.23$

   > I used trial and error to find the answer.

   On day 14 of the savings plan, the amount she deposited would exceed \$10,000.

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Lesson 6: Exponential Growth—U.S. Population and World Population

Stuart plans to deposit $1,000 into a savings account. His bank offers two different types of savings accounts. Option A pays a simple interest rate of 3.2% per year. Option B pays a compound interest rate of 2.8% per year, compounded monthly.

a. Write an explicit formula for the sequence that models the balance in Stuart's account t years after he deposits the money if he chooses option A.

$A(t) = 1000 + 1000(0.032)t$

> I know that simple interest means that the same amount of interest will be added each year. I can use the formula $I = Prt$ to write an expression for the total interest.

b. Write an explicit formula for the sequence that models the balance in Stuart's account m months after he deposits the money if he chooses option B.

$B(t) = 1000\left(1 + \frac{0.028}{12}\right)^m$

> Since the interest is compounded monthly, I need to divide the annual interest rate by 12 to find the interest rate per month.

c. Which option is represented with a linear model? Why?

Option A is represented with a linear model because there is a constant rate of change each year.

d. Which option is represented with an exponential model? Why?

Option B is represented with an exponential model because there is a constant ratio of change each month.

> I need to determine when $A(t)$ and $B(t)$ will equal \$2000.

e. Approximately how long will it take Stuart to double his money if he chooses option A? Option B?

$$A(t) = 1000 + 1000(0.032)t$$
$$2000 = 1000 + 1000(0.032)t$$
$$t = 31.25$$

> I can solve this equation for t.

$$B(t) = 1000\left(1 + \frac{0.028}{12}\right)^m$$
$$2000 = 1000\left(1 + \frac{0.028}{12}\right)^m$$
$$m \approx 297$$

> I need to use trial and error to find m.

If he chooses option A, it will take him 32 years to double his money. If he chooses option B, it will take him 24 years and 9 months to double his money.

f. How should Stuart decide between the two options?

If he is going to invest for a short amount of time (fewer than 10 years), he should choose option A. If he is going to invest for a long amount of time (10 years or longer), he should choose option B.

$$A(10) = 1000 + 1000(0.032)10 = 1320$$
$$B(120) = 1000\left(1 + \frac{0.028}{12}\right)^{120} \approx 1322.70$$

> At 10 years (120 months), the balance in the account for option B is larger than the balance in the account for option A.

Lesson 7: Exponential Decay

1. Since 1950, the population of Detroit has been decreasing. The population of Detroit (in millions) can be modeled by the following formula:

   $P(t) = 1.85(0.985)^t$, where t is the number of years since 1950.

   I can see that this is an exponential decay model because $b < 1$ in the formula $P(t) = a(b)^t$.

   a. According to the model, what was the population of Detroit in 1950?

   $P(0) = 1.85(0.985)^0 = 1.85$

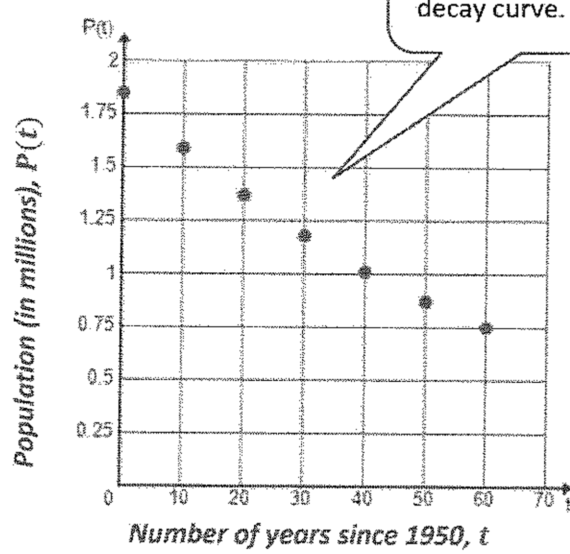
   In 1950, the population of Detroit was approximately 1.85 *million.*

   I need to find $P(0)$ since 1950 corresponds to $t = 0$.

   b. Complete the following table, and then graph the points $(t, P(t))$.

   I see that the points form an exponential decay curve.

t	$P(t)$
0	1.85
10	1.59
20	1.37
30	1.18
40	1.01
50	0.87
60	0.75



c. If this trend continues, estimate the year in which the population of Detroit will be less than 500,000.

I can use trial and error or the table feature on a graphing calculator to find the answer.

$P(87) = 1.85(0.985)^{87} \approx 0.497$

If this trend continues, the population of Detroit will be less than $500,000$ *by the year 2037.*

2. A Christmas tree farmer has 6,000 firs on his farm. Each Christmas, he plans to cut down 12% of his trees.

   a. Write a formula to model the number of trees on his farm each year.

   $N(t) = 6000(1 - 0.12)^t = 6000(0.88)^t$, *where t represents the number of years.*

   If the farmer cuts down 12% of the trees each year, then 88% of the trees are remaining.

   b. If he does not plant any new trees, how many trees will he have on his farm in 15 years?

   $N(15) = 6000(0.88)^{15} \approx 881.843$

   After 15 *years, the farmer will have approximately* 882 *trees on his farm.*

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Lesson 8: Why Stay with Whole Numbers?

Writing an Explicit Formula from a Table or Graph

1. The first four terms of sequence A are given in the table.

n	$A(n)$
1	5
2	10
3	20
4	40

a. Create an explicit formula for the sequence.

$A(n) = 5(2)^{n-1}$

> This sequence grows with a "times 2" pattern. I need to use the formula for a geometric sequence.

b. When will this sequence exceed 500? Explain how you know.

$A(5) = 80$, $A(6) = 160$, $A(7) = 320$, $A(8) = 640$

This sequence exceeds 500 by the 8th term.

> I can quickly double each term to find the first one that exceeds 500.

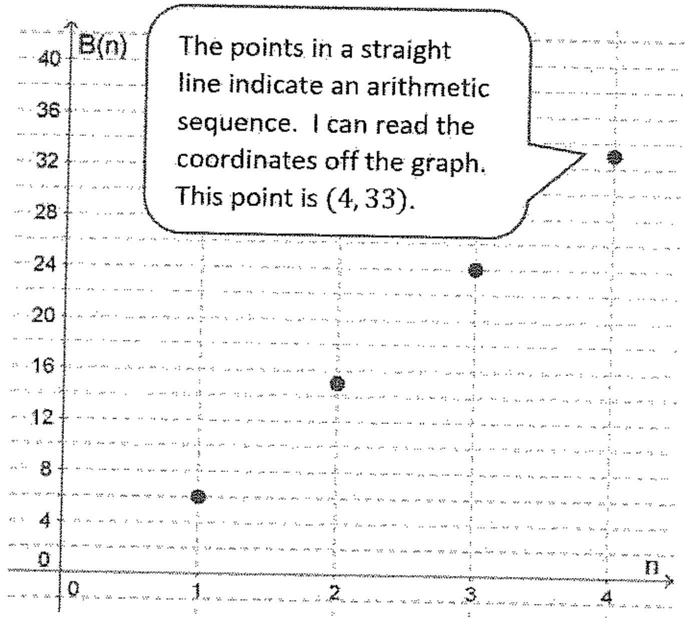
2. The first four terms of sequence B are graphed as a set of ordered pairs.

a. Create an explicit formula for the sequence.

$B(n) = 6 + 9(n - 1)$

b. Explain the meaning of the ordered pair $(3, 24)$.

It means that the third term in the sequence has a value of 24.



c. When will this sequence exceed 500? Explain how you know.

$$6 + 9(n - 1) > 500$$
$$9n - 9 > 494$$
$$9n > 503$$
$$n > \frac{503}{9}$$

I can make and solve an inequality, $B(n) > 500$.

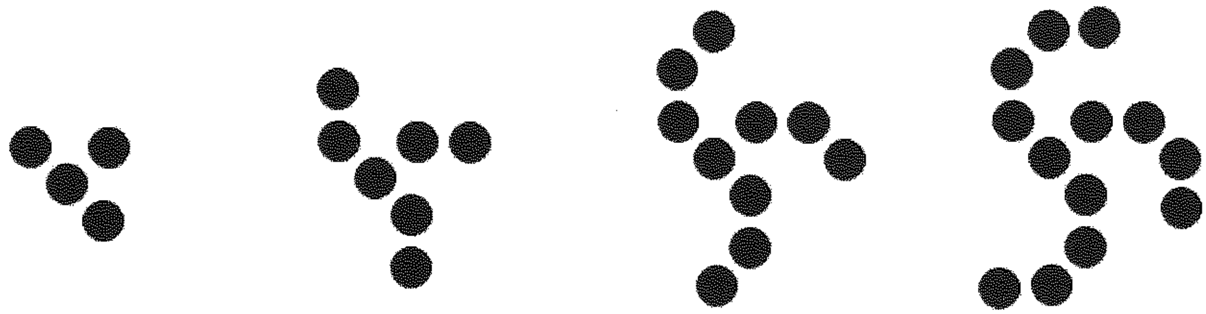
The solution is approximately 55.9. *The sequence will exceed* 500 *on the* 56^{th} *term.*

$B(56) = 6 + 9(56 - 1) = 501$

Writing an Explicit Formula for a Pattern

3. A dot pattern is shown below.

Figure 1 Figure 2 Figure 3 Figure 4



a. How is this pattern growing?

Each figure contains three more dots than the previous figure.

b. Create an explicit formula that could be used to determine the number of dots in the n^{th} figure.

$D(n) = 3n + 1$, *where* n *is a positive integer.*

The set of positive integers starts with 1.

This is an arithmetic sequence with a constant difference of 3. I know the formula must have a $3n$ term in it. I need the 1 to make the first term be equal to 4.

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c. Evaluate your formula for $n = 0$ and $n = \frac{5}{3}$. Draw Figure 0 and Figure $\frac{5}{3}$, and explain how you decided to create your drawings.

$D(0) = 3(0) + 1 = 1$, and $D\left(\frac{5}{3}\right) = 3\left(\frac{5}{3}\right) + 1 = 6$

You could draw one dot for Figure 0.

To draw Figure $\frac{5}{3}$, you could draw 6 dots, leaving off 1 dot from Figure 2.

Figure 0

Figure $\frac{5}{3}$

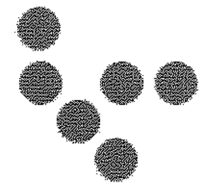


Figure 0 needs to have three fewer dots than Figure 1. Since $\frac{5}{3}$ is between 1 and 2, I know this figure will have between 4 and 7 dots.

Lesson 9: Representing, Naming, and Evaluating Functions

Lesson Notes

This lesson focuses students on understanding the definition of a function. Students are introduced to ways to represent functions that may differ from typical high school textbooks. This intentional choice allows students to understand functions at a deeper level that they will need in future mathematics courses. A function has three essential parts: (1) a set of domain values, (2) a set of range values, and (3) a method for assigning each element of the domain to exactly one element of the range.

Does a Correspondence Represent a Function?

Which of the following descriptions represents a function?

Justify your answers.

> The first part of the phrase identifies the domain and the second part the range. I know each domain element can only be assigned one range element.

1. The assignment of students in your school to a grade level

   Yes. This is an example of a function because each student is only assigned to one grade level based on the credits he has earned (or his year in school).

2. The assignment of students to textbooks at your school

   No. Most students in my school are assigned more than one textbook during any given school year.

3. The assignment of a person to a driver's license number in your state

   Yes. Each person that earns a driver's license is assigned only one number.

4. A non-negative real number to its square roots

   No. Each positive real number has two square roots. For example, the square roots of 4 are 2 and -2 because $(-2)^2 = 4$ and $(2)^2 = 4$.

> I learned about the definition of square root in Grade 8.

Sequences Are Functions

5. The sequence of even numbers is $\{2, 4, 6, 8, \ldots\}$. Here is a function that represents this sequence.

   Let f: {positive integers} → {even numbers}

   Assign each term number to the even number $2n$, where n is the term number.

   I know that $f(5)$ is the range element for the domain element 5.

   a. What is $f(5)$? What does it mean?

   $f(5) = 10$ because $2(5)$ is 10. This means that the fifth term in the sequence is 10.

   I know the range element is 22, and I want to find the term number that is the domain.

   b. What is the solution to the equation $f(n) = 22$? What is the meaning of this solution?

   The equation $2n = 22$ has the solution 11. This means that the 11^{th} term of the sequence is 22.

   c. According to this definition, is $\frac{1}{2}$ in the domain of f? Explain why or why not.

   No. The domain is positive integers, and $\frac{1}{2}$ is not an integer.

   d. According to this definition, is 55 in the range of f? Explain why or why not.

   No. The range must be an even number, and 55 is an odd number.

6. Write each sequence as a function.

   These numbers are all powers of 2.

   I need to reduce the term number because $2^0 = 1$ and the first term is 1

   a. $\{1, 2, 4, 8, 16, 32\}$

   Let f: $\{1, 2, 3, 4, 5, 6\} \rightarrow \{1, 2, 4, 8, 16, 32\}$

   Assign each term number to the number 2 raised to the integer 1 less than the term number.

   b. $a_{n+1} = a_n + 3$, $a_1 = 3$, where n is a positive integer greater than 1.

   Let f: {positive integers} → {positive integers}

   Assign to each positive integer the number that is 3 times the integer.

   I can write out the first few terms using the recursive definition to help me figure out the description of the function. $3, 6, 9, 12, \ldots$

Lesson 10: Representing, Naming, and Evaluating Functions

Evaluating Functions

Let $f(x) = 2x - 9$, and let $g(x) = -3\left(\frac{1}{2}\right)^x$. Find the value of each function for the given input.

1. $f(2)$

   $f(2) = 2(2) - 9$
   $= -5$

   The value of $f(2)$ is -5.

   > The input is the number or expression inside the parentheses. I need to substitute this number for x and evaluate the expression.

2. $f(\sqrt{5})$

   $f(\sqrt{5}) = 2(\sqrt{5}) - 9$
   ≈ -4.53

   The value of $f(\sqrt{5})$ is approximately 4.53.

3. $f(3) - f(-2)$

   > I must remember that these parentheses do not mean multiply. $f(3)$ means the range value for the domain input 3. Here, I need to subtract two range values.

   $f(3) = 2(3) - 9 = -3$
   $f(-2) = 2(-2) - 9 = -13$

   $f(3) - f(-2) = -3 - (-13)$
   $f(3) - f(-2) = 10$

4. $f(b)$

   $f(b) = 2b - 9$

   > A variable symbol is a placeholder for a number. I must substitute the variable or the expression into the formula for the function. I can rewrite the resulting expression using the properties of algebra.

5. $f(b + 2)$

   $f(b + 2) = 2(b + 2) - 9$
   $= 2b - 5$

6. $f(b+2) - f(2)$

$$f(b+2) - f(2) = 2b - 5 - (2 \cdot 2 - 9)$$
$$= 2b - 5 + 5$$
$$= 2b$$

I need to subtract the value of the function when x is 2 from my answer to Problem 5.

7. $g(0)$

Now I need to substitute the input value into the formula for function g.

$$g(0) = -3\left(\frac{1}{2}\right)^0$$
$$= -3(1)$$
$$= -3$$

The value of $g(0)$ is -3 or $g(0) = -3$.

8. $g(-3)$

$$g(-3) = -3\left(\frac{1}{2}\right)^{-3}$$
$$= -3(2)^3$$
$$= -24$$

I recall the definition of negative exponents from Grade 8: $\frac{1}{x^n} = x^{-n}$ for x not equal to 0.

The value of $g(-3)$ is -24.

9. $g(x+h) - g(x)$

First, find $g(x+h)$.

$$g(x+h) = -3\left(\frac{1}{2}\right)^{x+h}$$
$$= -3\left(\frac{1}{2}\right)^x \cdot \left(\frac{1}{2}\right)^h$$

Then, subtract $g(x)$ from this expression to get the expression for $g(x+h) - g(x)$.

$$-3\left(\frac{1}{2}\right)^x \cdot \left(\frac{1}{2}\right)^h - 3\left(\frac{1}{2}\right)^x = -3\left(\frac{1}{2}\right)^x\left(\left(\frac{1}{2}\right)^h - 1\right)$$
$$g(x+h) - g(x) = -3\left(\frac{1}{2}\right)^x\left(\left(\frac{1}{2}\right)^h - 1\right)$$

I can rewrite the expression using the properties of exponents and the distributive property.

Domain and Range of a Function

10. What is the range of each function?

a. $h(x) = 3^x$ for $x \leq 5$.

The range is $h(x) \in (0, 243]$.

This means x can be any number less than or equal to 5. I know that 3^x must be positive, so the range cannot include 0 or negative numbers.

b. $f(x) = x - 3$, where x is a positive integer.

The range is all integers greater than or equal to -2.

I can express the domain and range sets using words, inequalities, or interval notation.

11. Provide a suitable domain and range to complete the definition of each function.

a. $f(x) = 3x - 5$

Domain: all real numbers, Range: all real numbers

b. $h(x) = -16x^2 + 1600$, where $h(x)$ is the height of a ball in feet, and x is the time since the ball was dropped from a cliff for 10 seconds.

Domain: $x \in [0, 10]$, Range: $h(x) \in [0, 1600]$

I can start keeping time from $x = 0$. The largest value of h is when $x = 0$. The smallest value is when $x = 10$. The ball's height can be any real number between 0 ft. and 1600 ft.

Working with Functions

12. Let $f: X \to Y$, where X and Y are the set of all real numbers, and x and h are real numbers. Suppose $f(x) = x^3$.

I need to compare the values of $f(x + h)$ and $f(x) + f(h)$.

a. Show that $f(x + h) = f(x) + f(h)$ is not true when $x = 1$ and $h = 2$.

$f(1 + 2) = (1 + 2)^3 = 27$. $f(1) = 1^3 = 1$ *and* $f(2) = 2^3 = 8$. $27 \neq 1 + 8$.

b. Find a value for x and a value for h so that $f(x + h) = f(x) + f(h)$ is a true statement.

Let $x = 1$ *and* $h = 0$. $f(1 + 0) = (1 + 0)^3 = 1$. $f(1) = 1$ *and* $f(0) = 0$. $1 = 1 + 0$.

c. Explain why $f(x + h) = f(x) + f(h)$ is not true for all real numbers x and h.

The solution to part (a) is not a true statement. If the statement is not true for one pair of real numbers, then it cannot be true for all real numbers.

13. Given the function f whose domain is the set of real numbers, let $f(x) = 1$ if x is a positive number, let $f(x) = 0$ if x is 0, and let $f(x) = -1$ if x is a negative number.

a. Explain why f is a function.

I recall the definition of a function from Lesson 9.

Each real number in the domain set is only assigned to one real number in the range set.

b. What is the range of f?

The range is $\{-1, 0, 1\}$.

According to the definition of the function, there are only 3 range values.

c. Complete the table to find the value of the function for the given inputs.

x	π	$-\frac{2}{5}$	1,000,000	$\sqrt{4}-2$
$f(x)$	1	-1	1	0

I can evaluate the expression $\sqrt{4}-2$ to know what value to assign this input.

d. List 2 possible values for x if $f(x) = 1$.

x can be any positive number. 4 and 400 will work.

Lesson 11: The Graph of a Function

Lesson Notes

Students deepen their understanding of functions and the meaning of the graph of a function by connecting these ideas to computer programming. They evaluate functions, graph functions, and represent functions using set-builder notation.

Apply Computer Programming to Generate Function Values and Plot Them

Perform the instructions for each of the following programming codes as if you were a computer and your paper were the computer screen.

1.

```
Declare x integer
For all x from 0 to 5
        Print x^2 + 2x
Next x
```

I know this code prints a list of numbers. I need to evaluate the expression $x^2 + 2x$ for each x value.

The values of x are $0, 1, 2, 3, 4,$ and 5.

Value of x	*Work*	*Value To Be Printed*
0	$0^2 + 2(0) = 0 + 0$	0
1	$1^2 + 2(1) = 1 + 2$	3
2	$2^2 + 2(2) = 4 + 4$	8
3	$3^2 + 2(3) = 9 + 6$	15
4	$4^2 + 2(4) = 16 + 8$	24
5	$5^2 + 2(5) = 25 + 10$	35

A table can help me organize my work.

2.

```
Declare x integer
Let f(x) = 2^x + 1
Initialize G as {}
For all x from -2 to 3
        Append (x, f(x)) to G
Next x
Plot G
```

This code generates a set of ordered pairs that I need to plot in the coordinate plane.

The values of x are $-2, -1, 0, 1, 2,$ *and* 3.

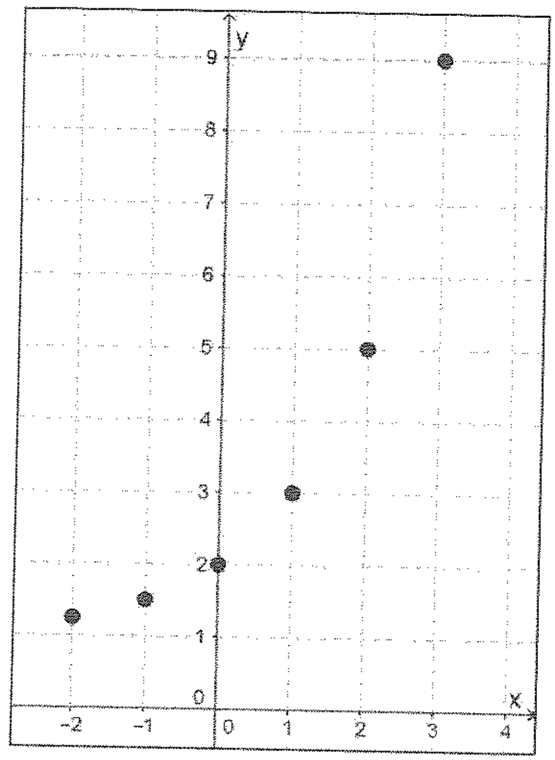
To find the $f(x)$ *value for each* x, *substitute each* x *into the expression* $2^x + 1$. *For example, when* $x = -2$,

$$2^{-2} + 1 = \frac{1}{2^2} + 1 = 1.25.$$

Thus, the ordered pair $(-2, 1.25)$ *is in set* G.

Use the ordered pairs in set G *to plot* G.

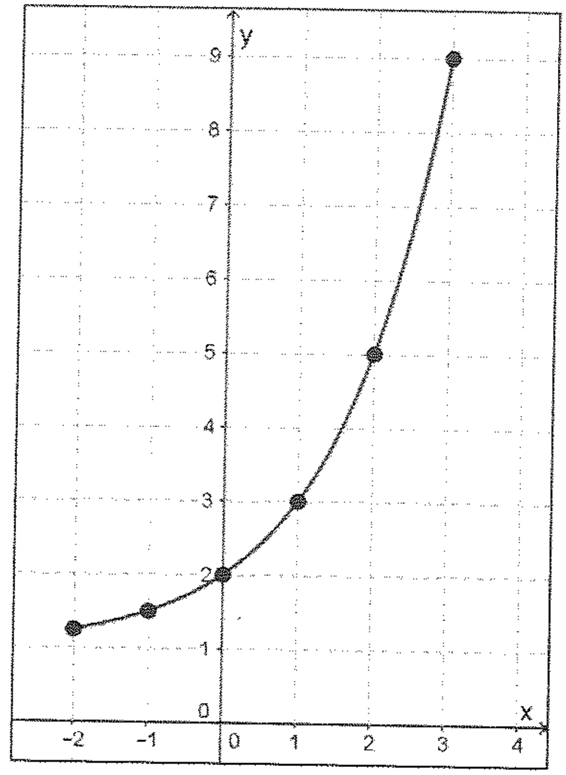
The ordered pairs in set G *are* $\{(-2, 1.25), (-1, 1.5), (0, 2), (1, 3), (2, 5), (3, 9)\}$.



3.

```
Declare x real
Let f(x) = 2^x + 1
Initialize G as {}
For all x such that −2 ≤ x ≤ 3
        Append (x, f(x)) to G
Next x
Plot G
```

There are infinitely many real numbers between −2 and 3. I can use integer values for x to generate enough ordered pairs to plot the entire graph by connecting those points in a smooth curve.



4. Compare and contrast the graphs of the functions plotted in Problems 2 and 3.

The functions have the same input/output pairs for the integers from -2 *to* 3*. However, the domain of the second function is all the real numbers from* -2 *to* 3*, so the graph represents all possible domain and range pairs by drawing it as a connected curve. The graph of the function in Problem 2 is just a set of* 5 *ordered pairs.*

5. Use the graph in Problem 3 to state the range of the function described by the programming code.

The range of the function is $1.25 \le f(x) \le 9$.

The Graph of a Function and Set-Builder Notation

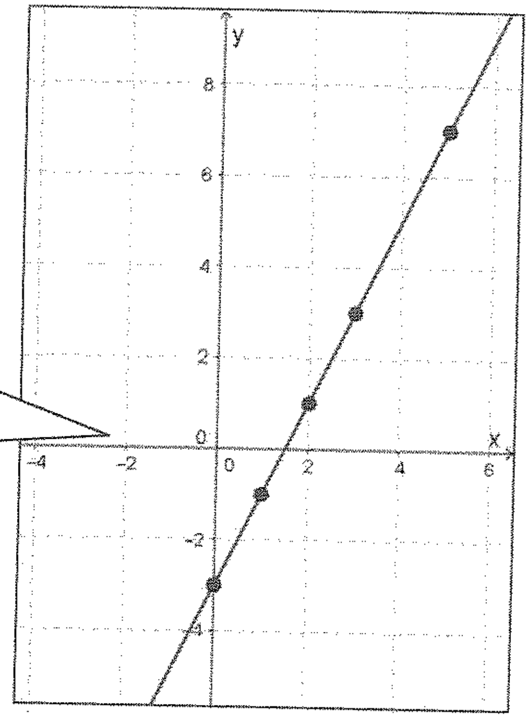
Sketch the graph of the functions defined by the following formulas, and write the graph of f as a set using set-builder notation. (Hint: Assume the domain is all real numbers unless specified in the problem.)

6. $f(x) = 2x - 3$

The graph of $f = \{(x, 2x - 3) \mid x \text{ real}\}$.

The format for writing set-builder notation is in the Lesson 11 Summary.

To construct the graph of f, I can select several integer values of x and find the corresponding $f(x)$ value by evaluating the function at each x. After plotting these ordered pairs, I can connect them with a smooth curve. I could organize my work in a table like I did in Problem 1. The domain is all real numbers, so I need to extend my graph to fill the grid.



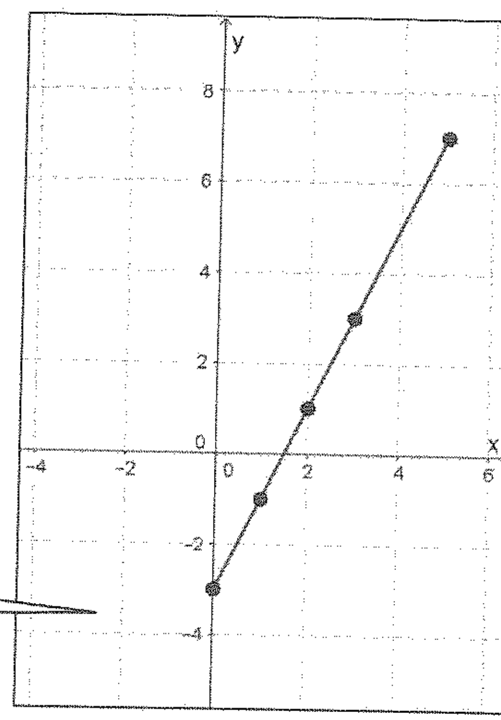
7. $f(x) = (x-2)(x+4) - (x^2 - 5)$ for $0 \le x \le 5$.

Rewrite $(x-2)(x+4) - (x^2 - 5)$ *using the properties of algebra.*

$$\begin{aligned} f(x) &= (x-2)(x) + (x-2)(4) - x^2 + 5 \\ &= x^2 - 2x + 4x - 8 - x^2 + 5 \\ &= 2x - 3 \end{aligned}$$

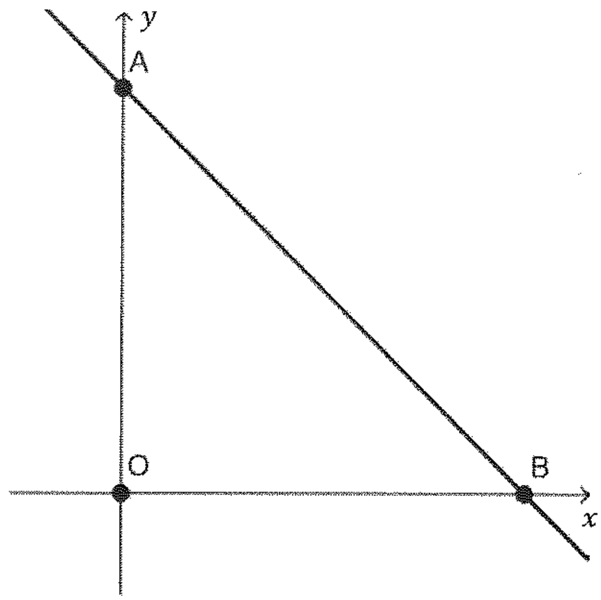
The graph of $f = \{(x, 2x - 3) \mid x \text{ real}, 0 \le x \le 5\}$.

I can only select values of x from 0 to 5 according to the function's domain.



Working with the Graph of a Function

8. The graph of the function $f(x) = 5 - x$ is shown to the right.



a. What are the coordinates of point A and point B? Explain how you know.

Point A is the y-intercept point. This point has coordinates $(0, f(0))$.

Evaluate $f(0)$.

$$\begin{aligned} f(0) &= 5 - 0 \\ &= 5 \end{aligned}$$

The coordinates of point A are $(0, 5)$.

Point B is the x-intercept point. This point has coordinates $(x, 0)$ for some real x.

Set $f(x)$ equal to zero to find this value of x.

$5 - x = 0$ when $x = 5$.

The coordinates of point B are $(5, 0)$.

b. What is the length of line segment AB?

Triangle AOB is a right triangle. Use the Pythagorean theorem to find AB.

$$\begin{aligned} OA^2 + OB^2 &= AB^2 \\ 5^2 + 5^2 &= AB^2 \\ 50 &= AB^2 \end{aligned}$$

I learned the Pythagorean theorem in Grade 8. It relates the lengths of the legs of a right triangle, a and b, to the hypotenuse c with the formula $a^2 + b^2 = c^2$.

The positive solution to this equation is $\sqrt{50}$ because $50 = \left(\sqrt{50}\right)^2$.

Segment AB is $\sqrt{50}$ units long.

Lesson 12: The Graph of the Equation $y = f(x)$

Lesson Notes

Students understand the difference between the graph of the function f and the graph of the equation $y = f(x)$. This difference is subtle and often overlooked because the two different processes produce the exact same graph.

Apply Computer Programming to Generate Function Values and Plot Them

Perform the instructions for each of the following programming codes as if you were a computer and your paper were the computer screen.

I know this code asks me to test whether or not the equation is true for the given values of x.

1.

```
Declare x integer
For all x from 0 to 5
        If (x − 2)^2 = 4 then
                Print True
        else
                Print False
        End if
Next x
```

The values of x are $0, 1, 2, 3, 4$, and 5.

Value of x	*Does $(x-2)^2 = 4$?*	*Print*
0	$(0-2)^2 = (-2)^2 = 4$	*True*
1	$(1-2)^2 = (-1)^2 = 1$	*False*
2	$(2-2)^2 = (0)^2 = 0$	*False*
3	$(3-2)^2 = (1)^2 = 1$	*False*
4	$(4-2)^2 = (2)^2 = 4$	*True*
5	$(5-2)^2 = (3)^2 = 9$	*False*

A table can help me organize my work.

2.

```
Declare x integer
Initialize G as {}
For all x from 0 to 5
        If (x − 2)^2 = 4 then
                Append x to G
         else
                Do NOT append x to G
        End if
Next x
Print G
```

$G = \{0, 4\}$

This code checks to see if the integers from 0 to 5 are solutions to the equation. If an integer value is a solution, it assigns that number to set G and prints it.

The equation is the same as Problem 1. I can just use the values of x above that gave a true statement.

The Graph of an Equation and Set-Builder Notation

3. Use the thought code below to answer the following questions.

```
Declare x and y real
Let f(x) = (x − 2)^2
Initialize G as {}
For all x in the real numbers
        For all y in the real numbers
                If y = f(x) then
                        Append (x, y) to G
                 else
                        Do NOT append (x, y) to
G
                End if
        Next y
Next x
Plot G
```

I know this thought code takes each real number x and checks to see which real number y makes $y = f(x)$ a true statement. It assigns that ordered pair (x, y) to set G and plots the solutions to $y = f(x)$ by plotting all the ordered pairs in G.

a. What is the domain of f?

The domain is the set of all real numbers.

b. What is the range of f?

The range is the set of all non-negative real numbers, thus $[0, \infty)$.

I can graph f using a graphing calculator, or I can create a table of values showing a few of the (x, y) pairs that belong to set G and then plot those points.

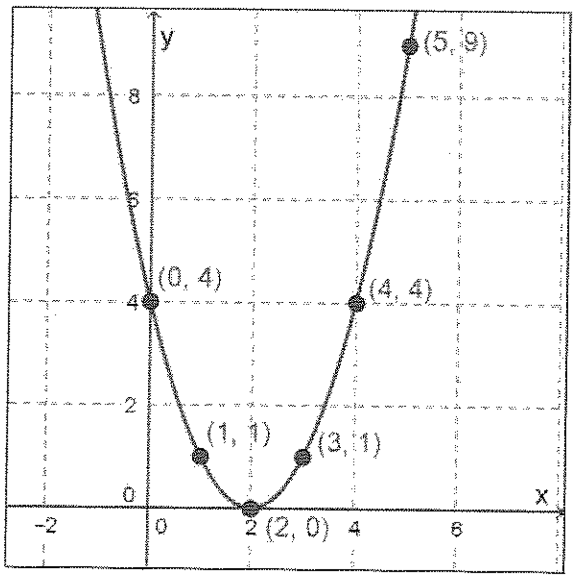
c. Write the set G using set-builder notation.

$\{(x, y) \mid y = (x - 2)^2\}$

d. Plot the set G to obtain the graph of the function $f(x) = (x - 2)^2$.

x	$y = (x-2)^2$
0	4
1	1
2	0
3	1
4	4
5	9

I need to select enough x-values to see the shape of the graph. I know this is not a linear function because of the exponent.



Writing Thought Code Given Set-Builder Notation

Answer the following questions about the set:

$$\{(x, y) \mid -3 \le x \le 3 \text{ and } y = 2x - 3\}.$$

4. The equation can be rewritten in the form $y = f(x)$, where $f(x) = 2x - 3$.
   a. What are the domain and range of the function f specified by the set?

      Domain: $-3 \le x \le 3$

      Range: $-9 \le y \le 3$ *or* $-9 \le f(x) \le 3$ *for x in the domain*

I know this graph is a line with a positive slope. Therefore, $f(-3)$ will be the lowest range value, and $f(3)$ will be the highest.

b. Write thought code such as that in Problem 3 that will generate, and then plot the set.

Declare x and y real
Let $f(x) = 2x - 3$
Initialize G as {}
For all x such that $-3 \le x \le 3$
For all y such that $-9 \le y \le 3$
If $y = f(x)$ then
Append (x, y) to G
else
Do NOT append (x, y) to G
End if
Next y
Next x
Plot G

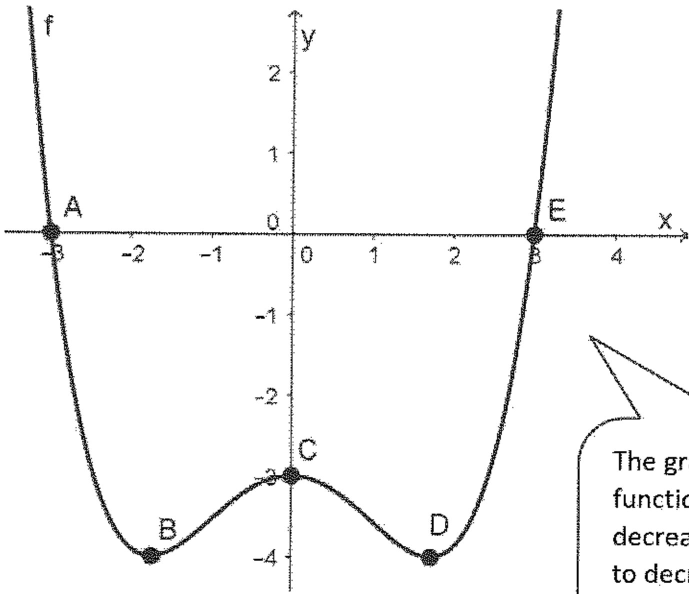
I can use the sample in Exploratory Challenge 1 or Problem 3 above and just change the function, domain, and range. Since all equations of the form $y = f(x)$ are plotted the same way, I don't need to change anything else.

Increasing and Decreasing Intervals and Relative Maximum and Relative Minimum Points

5. The function $f(x) = 2x - 3$ is an increasing function on the domain all real numbers. Show that the function satisfies the definition of increasing for the points 3 and 5 on the number line; that is, show that since $3 < 5$, then $f(3) < f(5)$.

   $f(3) = 3$ and $f(5) = 7$. Since $3 < 7$, $f(3) < f(5)$. This satisfies the definition of increasing.

6. Answer the following about the graph of a function below.



The graph of this function changes from decreasing to increasing to decreasing to increasing as x gets larger.

a. Which points (A, B, C, D, or E) are relative maxima?

C

I can refer to the student pages of this lesson to recall the meaning of the vocabulary words.

b. Which points (A, B, C, D, or E) are relative minima?

B and D

c. Name any interval where the function is increasing.

$[2, 3]$ *or* $[-1, 0]$ *would work.*

d. Name any interval where the function is decreasing.

$[-3, -2]$ *or* $[0, 1]$ *would work.*

I can only estimate these intervals because I don't know the exact coordinates of the points.

Always use domain (x-values) to describe increasing and decreasing intervals.